

## Solutions to Exercises (Sections 1.1 - 1.10)

### Section 1.1

Exercise 1.1.1: Identifying propositions

(a) Have a nice day.

Solution: Command, not a proposition.

(b) The soup is cold.

Solution: Proposition. Negation: The soup is not cold.

(c) The patient has diabetes.

Solution: Proposition. Negation: The patient does not have diabetes.

(d) The light is on.

Solution: Proposition. Negation: The light is off.

(e) It's a beautiful day.

Solution: Proposition. Negation: It is not a beautiful day.

(f) Do you like my new shoes?

Solution: Question, not a proposition.

(g) The sky is purple.

Solution: Proposition. Negation: The sky is not purple.

(h)  $2 + 3 = 6$

Solution: Proposition. Negation:  $2 + 3 \neq 6$ .

(i) Every prime number is even.

Solution: Proposition. Negation: It is not true that every prime number is even.

(j) There is a number that is larger than 17.

Solution: Proposition. Negation: There is no number that is larger than 17.

Exercise 1.1.2: Expressing English sentences using logical notation

(a) The patient had nausea and migraines.

Solution:  $n \wedge m$

(b) The patient took the medication, but still had migraines.

Solution:  $t \wedge m$

(c) The patient had nausea or migraines.

Solution:  $n \vee m$

(d) The patient did not have migraines.

Solution:  $\neg m$

(e) Despite the fact that the patient took the medication, the patient had nausea.

Solution:  $t \wedge n$

(f) There is no way that the patient took the medication.

Solution:  $\neg t$

### Exercise 1.1.3: Applying logical operations

(a)  $\neg p$

Solution: True

(b)  $p \vee r$

Solution: False

(c)  $q \wedge s$

Solution: True

(d)  $q \vee s$

Solution: True

(e)  $q \oplus s$

Solution: False

(f)  $q \oplus r$

Solution: True

### Exercise 1.1.4: Truth values for statements with inclusive and exclusive or

(a) February has 31 days or the number 5 is an integer.

Solution: Inclusive or: True. Exclusive or: True.

(b) The number  $\pi$  is an integer or the sun revolves around the earth.

Solution: Inclusive or: False. Exclusive or: False.

(c) 20 nickels are worth one dollar or whales are mammals.

Solution: Inclusive or: True. Exclusive or: False.

(d) There are eight days in a week or there are seven days in a week.

Solution: Inclusive or: True. Exclusive or: True.

(e) January has exactly 31 days or April has exactly 30 days.

Solution: Inclusive or: True. Exclusive or: False.

## Section 1.2

### Exercise 1.2.1: Truth values for compound English sentences

(a) 5 is an odd number and 3 is a negative number.

Solution: Since 3 is not a negative number, the proposition is false.

(b) 5 is an odd number or 3 is a negative number.

Solution: 5 is an odd number, so the proposition is true.

(c) 8 is an odd number or 4 is not an odd number.

Solution: 4 is not an odd number, so the proposition is true.

(d) 6 is an even number and 7 is odd or negative.

Solution: 7 is odd and 6 is even, so the proposition is true.

(e) It is not true that 7 is an odd number or 8 is an even number.

Solution: It is true that 7 is odd or 8 is even, so the proposition is false.

Exercise 1.2.2: Truth values for compound propositions

(a)  $p \vee \neg q$

Solution: True.

(b)  $(p \wedge q) \vee s$

Solution: True.

(c)  $p \wedge (q \vee s)$

Solution: True.

(d)  $p \wedge \neg(q \vee s)$

Solution: False.

(e)  $\neg(q \wedge p \wedge \neg s)$

Solution: False.

(f)  $\neg(p \wedge \neg(q \wedge s))$

Solution: False.

Exercise 1.2.3: Translating compound propositions into English sentences

(a)  $\neg p$

Solution: I am not going to a movie tonight.

(b)  $p \wedge q$

Solution: I am going to a movie and the party tonight.

(c)  $p \wedge \neg q$

Solution: I am going to a movie and not the party tonight.

(d)  $\neg p \vee \neg q$

Solution: I am either not going to a movie or not going to the party (or neither) tonight.

(e)  $\neg(p \wedge q)$

Solution: It is not true that I am going to a movie and the party tonight.

Exercise 1.2.4: Multiple disjunction or conjunction operations

Suppose that  $p$ ,  $q$ ,  $r$ ,  $s$ , and  $t$  are all propositional variables.

(a) Describe in words when the expression  $p \vee q \vee r \vee s \vee t$  is true and when it is false.

Solution: The expression is true if at least one of the propositional variables is true. The expression is false if all of the propositional variables are false.

(b) Describe in words when the expression  $p \wedge q \wedge r \wedge s \wedge t$  is true and when it is false.

Solution: The expression is false if at least one of the propositional variables is false. The expression is true if all of the propositional variables are true.

### Exercise 1.2.5: Expressing a set of conditions using logical operations

Consider the following pieces of identification a person might have in order to apply for a credit card:

B: Applicant presents a birth certificate.

D: Applicant presents a driver's license.

M: Applicant presents a marriage license.

Write a logical expression for the requirements under the following conditions:

(a) The applicant must present either a birth certificate, a driver's license or a marriage license.

Solution:  $B \vee D \vee M$

(b) The applicant must present at least two of the following forms of identification: birth certificate, driver's license, marriage license.

Solution:  $(B \wedge M) \vee (B \wedge D) \vee (M \wedge D)$

(c) Applicant must present either a birth certificate or both a driver's license and a marriage license.

Solution:  $B \vee (D \wedge M)$

### Exercise 1.2.6: Boolean expression to express a condition on the input variables

Give a logical expression with variables  $p$ ,  $q$ , and  $r$  that is true if  $p$  and  $q$  are false and  $r$  is true and is otherwise false.

Solution:  $\neg p \wedge \neg q \wedge r$

## Section 1.3

### Exercise 1.3.1: Truth values for conditional statements in English

(a) If February has 30 days, then 7 is an odd number.

Solution: True. The hypothesis is false and the conclusion is true.

(b) If January has 31 days, then 7 is an even number.

Solution: False. The hypothesis is true but the conclusion is false.

(c) If 7 is an odd number, then February does not have 30 days.

Solution: True. The hypothesis and conclusion are both true.

(d) If 7 is an even number, then January has exactly 28 days.

Solution: True. The hypothesis and conclusion are both false.

Exercise 1.3.2:

Give the inverse, converse and contrapositive for each of the following statements:

(a) If she finished her homework, then she went to the party.

Solution

Inverse: If she did not finish her homework, then she did not go to the party.

Contrapositive: If she did not go to the party, then she did not finish her homework.

Converse: If she went to the party, then she finished her homework.

(b) If he trained for the race, then he finished the race.

Solution

Inverse: If he did not train for the race, then he did not finish the race.

Contrapositive: If he did not finish the race, then he did not train for the race.

Converse: If he finished the race, then he trained for the race.

(c) If the patient took the medicine, then she had side effects.

Solution

Inverse: If the patient didn't take the medicine, then she didn't have side effects.

Contrapositive: If the patient didn't have side effects, then she didn't take the medicine.

Converse: If the patient had side effects, then she took the medicine.

Exercise 1.3.3: Expressing conditional statements in English using logic

Define the following propositions:

c: I will return to college.

j: I will get a job.

Translate the following English sentences into logical expressions using the definitions above:

(a) Not getting a job is a sufficient condition for me to return to college.

Solution:  $\neg j \rightarrow c$

(b) If I return to college, then I won't get a job.

Solution:  $c \rightarrow \neg j$

(c) I am not getting a job, but I am still not returning to college.

Solution:  $\neg j \wedge \neg c$

(d) I will return to college only if I won't get a job.

Solution:  $c \rightarrow \neg j$

(e) There's no way I am returning to college.

Solution:  $\neg c$

(f) I will get a job and return to college.

Solution:  $j \wedge c$

Exercise 1.3.4: Expressing conditional statements in English using logic

Define the following propositions:

s: a person is a senior

y: a person is at least 17 years of age

p: a person is allowed to park in the school parking lot

Express each of the following English sentences with a logical expression:

(a) A person is allowed to park in the school parking lot only if they are a senior and at least seventeen years of age.

Solution:  $p \rightarrow (s \wedge y)$

(b) A person can park in the school parking lot if they are a senior or at least seventeen years of age.

Solution:  $(s \vee y) \rightarrow p$

(c) Being 17 years of age is a necessary condition for being able to park in the school parking lot.

Solution:  $p \rightarrow y$

(d) A person can park in the school parking lot if and only if the person is a senior and at least 17 years of age.

Solution:  $p \leftrightarrow (s \wedge y)$

(e) Being able to park in the school parking lot implies that the person is either a senior or at least 17 years old.

Solution:  $p \rightarrow (s \vee y)$

Exercise 1.3.5: Finding logical expressions to match a truth table

For each table, give a logical expression whose truth table is the same as the one given.

(a)

p	q	?
T	T	F
T	F	T
F	T	F
F	F	F

Solution

$\neg(p \rightarrow q)$

The solution is not unique.

(b)

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

Solution

$$p \leftrightarrow \neg q$$

The solution is not unique.

## Section 1.4

### Exercise 1.4.1

Show whether each logical expression is a tautology, contradiction or neither.

(a)  $(p \vee q) \vee (q \rightarrow p)$

Solution

Tautology. Every truth value in the column for  $(p \vee q) \vee (q \rightarrow p)$  is T.

p	q	$(p \vee q) \vee (q \rightarrow p)$
T	T	T
T	F	T
F	T	T
F	F	T

(b)  $(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$

Solution

Contradiction. Every truth value in the column for  $(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$  is F.

p	q	$(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$
T	T	F
T	F	F
F	T	F
F	F	F

$$(c) (p \rightarrow q) \leftrightarrow p$$

Solution

Neither. If  $p = T$  and  $q = T$ , then  $(p \rightarrow q) \leftrightarrow p$  is true. For any other truth assignment,  $(p \rightarrow q) \leftrightarrow p$  is false.

$$(d) (p \rightarrow q) \vee p$$

Solution

Tautology. Every truth value in the column for  $(p \rightarrow q) \vee p$  is T.

p	q	$(p \rightarrow q) \vee p$
T	T	T
T	F	T
F	T	T
F	F	T

$$(e) (\neg p \vee q) \leftrightarrow (p \wedge \neg q)$$

Solution

Contradiction. Every truth value in the column for  $(\neg p \vee q) \leftrightarrow (p \wedge \neg q)$  is F.

p	q	$(\neg p \vee q) \leftrightarrow (p \wedge \neg q)$
T	T	F
T	F	F
F	T	F
F	F	F

$$(f) (\neg p \vee q) \leftrightarrow (\neg p \wedge q)$$

Solution

Neither. If  $p = F$  and  $q = T$ , then  $(\neg p \vee q) \leftrightarrow (\neg p \wedge q)$  is true. If  $p = T$  and  $q = T$ , then  $(\neg p \vee q) \leftrightarrow (\neg p \wedge q)$  is false.

#### Exercise 1.4.2

Use truth tables to show that the following pairs of expressions are logically equivalent.

$$(a) p \leftrightarrow q \text{ and } (p \rightarrow q) \wedge (q \rightarrow p)$$



Solution

p	q	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

The columns for  $p \leftrightarrow q$  and  $(p \rightarrow q) \wedge (q \rightarrow p)$  are the same.

(b)  $\neg(p \leftrightarrow q)$  and  $\neg p \leftrightarrow q$

Solution

p	q	$\neg(p \leftrightarrow q)$	$\neg p \leftrightarrow q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

The columns for  $\neg(p \leftrightarrow q)$  and  $\neg p \leftrightarrow q$  are the same.

(c)  $\neg p \rightarrow q$  and  $p \vee q$

Solution

p	q	$\neg p \rightarrow q$	$p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The columns for  $\neg p \rightarrow q$  and  $p \vee q$  are the same.

### Exercise 1.4.3

Prove that the following pairs of expressions are not logically equivalent.

(a)  $p \rightarrow q$  and  $q \rightarrow p$

Solution: If  $p = T$  and  $q = F$ , then  $p \rightarrow q$  is false and  $q \rightarrow p$  is true. Also, when  $p = F$  and  $q = T$ , then  $p \rightarrow q$  is true and  $q \rightarrow p$  is false.

(b)  $\neg p \rightarrow q$  and  $\neg p \vee q$

Solution: If  $p = T$  and  $q = F$ , then  $\neg p \rightarrow q$  is true and  $\neg p \vee q$  is false. Also, when  $p = F$  and  $q = F$ , then  $\neg p \rightarrow q$  is false and  $\neg p \vee q$  is true.

(c)  $(p \rightarrow q) \wedge (r \rightarrow q)$  and  $(p \wedge r) \rightarrow q$

Solution: When  $p = T$ ,  $r = F$ , and  $q = F$  (or when  $p = F$ ,  $r = T$ , and  $q = F$ ), then  $(p \wedge r) \rightarrow q$  is true and  $(p \rightarrow q) \wedge (r \rightarrow q)$  is false.

(d)  $p \wedge (p \rightarrow q)$  and  $p \vee q$

Solution: When  $p = T$  and  $q = F$  (or when  $p = F$  and  $q = T$ ), then  $p \wedge (p \rightarrow q)$  is false and  $p \vee q$  is true.

#### Exercise 1.4.4

Determine whether the following pairs of expressions are logically equivalent. Prove your answer. If the pair is logically equivalent, then use a truth table to prove your answer.

(a)  $\neg(p \vee \neg q)$  and  $\neg p \wedge q$

Solution

Logically equivalent. The columns for  $\neg(p \vee \neg q)$  and  $\neg p \wedge q$  are the same.

$p$	$q$	$\neg(p \vee \neg q)$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	F	F

(b)  $\neg(p \vee \neg q)$  and  $\neg p \wedge \neg q$

Solution

Not logically equivalent. When  $p = F$  and  $q = F$ , then  $\neg(p \vee \neg q)$  is false and  $\neg p \wedge \neg q$  is true. Also, when  $p = F$  and  $q = T$ , then  $\neg(p \vee \neg q)$  is true and  $\neg p \wedge \neg q$  is false.

(c)  $p \wedge (p \rightarrow q)$  and  $p \rightarrow q$

Solution

Not logically equivalent. When  $p = F$  and  $q = F$  or  $T$ , then  $p \wedge (p \rightarrow q)$  is false and  $p \rightarrow q$  is true.

(d)  $p \wedge (p \rightarrow q)$  and  $p \wedge q$

Solution

Logically equivalent. The columns for  $p \wedge (p \rightarrow q)$  and  $p \wedge q$  are the same.

$p$	$q$	$p \wedge (p \rightarrow q)$	$p \wedge q$
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T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

#### Exercise 1.4.5

Translate each English sentence into a logical expression using the propositional variables defined below. Then negate the entire logical expression using parentheses and the negation operation. Apply De Morgan's law to the resulting expression and translate the final logical expression back into English.

p: the applicant has written permission from his parents

e: the applicant is at least 18 years old

s: the applicant is at least 16 years old

(a) The applicant has written permission from his parents and is at least 16 years old.

Solution

$p \wedge s$

$\neg(p \wedge s)$

$\neg p \vee \neg s$

The applicant does not have written permission from his parents or is not at least 16 years old.

(b) The applicant has written permission from his parents or is at least 18 years old.

Solution

$p \vee e$

$\neg(p \vee e)$

$\neg p \wedge \neg e$

The applicant does not have written permission from his parents and is not at least 18 years old.

### Section 1.5

#### Exercise 1.5.1

Predicates P, T, and E are defined below. The domain of discourse is the set of all positive integers.

$P(x)$ :  $x$  is even

$T(x, y)$ :  $2x = y$

$E(x, y, z)$ :  $xy = z$

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a)  $P(3)$

Solution: Proposition.  $P(3)$  is false because 3 is not even.

(b)  $\neg P(3)$

Solution: Proposition.  $\neg P(3)$  is true because  $P(3)$  is false.

(c)  $T(5, 32)$

Solution: Proposition.  $T(5, 32)$  is true because  $25 = 32$ .

(d)  $T(5, x)$

Solution: Not a proposition because  $x$  is a variable.

(e)  $E(6, 2, 36)$

Solution: Proposition.  $E(6, 2, 36)$  is true because  $6 \cdot 2 = 36$ .

(f)  $E(2, y, 7)$

Solution: Not a proposition because  $y$  is a variable.

(g)  $P(3) \vee T(5, 32)$

Solution: Proposition.  $P(3) \vee T(5, 32)$  is true because  $T(5, 32)$  is true.

(h)  $T(5, 16) \rightarrow E(6, 3, 36)$

Solution: Proposition.  $T(5, 16) \rightarrow E(6, 3, 36)$  is true because the hypothesis of the conditional operation,  $T(5, 16)$ , is false.

### Exercise 1.5.2

In this problem, the domain of discourse is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

(a)  $\exists x (x + x = 1)$

Solution: False.

(b)  $\exists x (x + 2 = 1)$

Solution: True. Example:  $x = -1$ .

(c)  $\forall x (x^2 - x \neq 1)$

Solution: True.

(d)  $\forall x (x^2 - x \neq 0)$

Solution: False. Counterexample:  $x = 1$  or  $x = 0$ .

(e)  $\forall x (x^2 > 0)$

Solution: False. Counterexample:  $x = 0$ .

(f)  $\exists x (x^2 > 0)$

Solution: True. Example: any integer besides 0.

### Exercise 1.5.3

Consider the following statements in English. Write a logical expression with the same meaning. The domain of discourse is the set of all real numbers.

(a) There is a number whose cube is equal to 2.

Solution:  $\exists x (x^3 = 2)$

(b) The square of every number is at least 0.

Solution:  $\forall x (x^2 \geq 0)$

(c) There is a number that is equal to its square.

Solution:  $\exists x (x^2 = x)$

(d) Every number is less than or equal to its square.

Solution:  $\forall x (x \leq x^2)$

### Exercise 1.5.4

The domain for this problem is a set  $\{a, b, c, d\}$ . The table below shows the value of three predicates for each of the elements in the domain. For example,  $Q(b)$  is false because the truth value in row b, column Q is F.

	P	Q	R
a	T	T	F
b	T	F	F
c	T	F	F
d	T	F	F

Which statements are true? Justify your answer.

(a)  $\forall x P(x)$

Solution: True.  $P(a)$ ,  $P(b)$ ,  $P(c)$ , and  $P(d)$  are all true.

(b)  $\exists x P(x)$

Solution: True. Elements a, b, c, and d are all examples.

(c)  $\forall x Q(x)$

Solution: False. Elements b, c, and d are all counterexamples.

(d)  $\exists x Q(x)$

Solution: True. The element a is an example.

(e)  $\forall x R(x)$

Solution: False. Elements a, b, c, and d are all counterexamples.

(f)  $\exists x R(x)$

Solution: False.  $R(a)$ ,  $R(b)$ ,  $R(c)$ , and  $R(d)$  are all false.

## Section 1.6

### Exercise 1.6.1

In the following question, the domain of discourse is the set of employees at a company. One of the employees is named Sam. Define the following predicates:

$T(x)$ : x is a member of the executive team

$B(x)$ : x received a large bonus

Translate the following English statements into a logical expression with the same meaning.

(a) Someone did not get a large bonus.

Solution:  $\exists x \neg B(x)$

(b) Everyone got a large bonus.

Solution:  $\forall x B(x)$

(c) Sam did not get a large bonus even though he is a member of the executive team.

Solution:  $\neg B(\text{Sam}) \wedge T(\text{Sam})$

(d) Someone who is not on the executive team received a large bonus.

Solution:  $\exists x (\neg T(x) \wedge B(x))$

(e) Every executive team member got a large bonus.

Solution:  $\forall x (T(x) \rightarrow B(x))$

### Exercise 1.6.2

Predicates P and Q are defined below. The domain of discourse is the set of all positive integers.

$P(x)$ : x is prime

$Q(x)$ : x is a perfect square (i.e.,  $x = y^2$ , for some integer y)

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a)  $\exists x Q(x)$

Solution: Proposition. True.

(b)  $\forall x Q(x) \wedge \neg P(x)$

Solution: Not a proposition because the variable x in  $P(x)$  is not bound by the quantifier.

(c)  $\forall x Q(x) \vee P(3)$

Solution: Proposition. The proposition is true.  $\forall x Q(x)$  is false because not every positive integer is a perfect square, but  $P(3)$  is true because 3 is prime. Therefore  $\forall x Q(x) \vee P(3)$  is true.

(d)  $\exists x (Q(x) \wedge P(x))$

Solution: Proposition. The proposition is false because there is no positive integer that is a perfect square and prime.

(e)  $\forall x (\neg Q(x) \vee P(x))$

Solution: Proposition. The proposition is false. One possible counterexample is  $x = 4$ .

### Exercise 1.6.3

In the following question, the domain of discourse is a set of students at a university. Define the following predicates:

$E(x)$ :  $x$  is enrolled in the class

$T(x)$ :  $x$  took the test

Translate the following English statements into a logical expression with the same meaning.

(a) Someone took the test who is enrolled in the class.

Solution:  $\exists x (T(x) \wedge E(x))$

(b) All students enrolled in the class took the test.

Solution:  $\forall x (E(x) \rightarrow T(x))$

(c) Everyone who took the test is enrolled in the class.

Solution:  $\forall x (T(x) \rightarrow E(x))$

(d) At least one student who is enrolled in the class did not take the test.

Solution:  $\exists x (E(x) \wedge \neg T(x))$

### Exercise 1.6.4

In the following question, the domain of discourse is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

$S(x)$ :  $x$  was sick yesterday

$W(x)$ :  $x$  went to work yesterday

$V(x)$ :  $x$  was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

(a) At least one person was sick yesterday.

Solution:  $\exists x S(x)$

(b) Everyone was well and went to work yesterday.

Solution:  $\forall x (\neg S(x) \wedge W(x))$

(c) Everyone who was sick yesterday did not go to work.

Solution:  $\forall x (S(x) \rightarrow \neg W(x))$

(d) Yesterday someone was sick and went to work.

Solution:  $\exists x (S(x) \wedge W(x))$

(e) Everyone who did not go to work yesterday was sick.

Solution:  $\forall x (\neg W(x) \rightarrow S(x))$

(f) Everyone who missed work was sick or on vacation (or both).

Solution:  $\forall x (\neg W(x) \rightarrow (S(x) \vee V(x)))$

(g) Someone who missed work was neither sick nor on vacation.

Solution:  $\exists x (\neg W(x) \wedge \neg S(x) \wedge \neg V(x))$

(h) Each person missed work only if they were sick or on vacation (or both).

Solution:  $\forall x (\neg W(x) \rightarrow (S(x) \vee V(x)))$

(i) Ingrid was sick yesterday but she went to work anyway.

Solution:  $S(\text{Ingrid}) \wedge W(\text{Ingrid})$

(j) Someone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression  $(x \neq \text{Ingrid})$ .)

Solution:  $\exists x ((x \neq \text{Ingrid}) \wedge S(x))$

(k) Everyone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression  $(x \neq \text{Ingrid})$ .)

Solution:  $\forall x ((x \neq \text{Ingrid}) \rightarrow S(x))$

### Exercise 1.6.5

In the following question, the domain of discourse is the set of employees of a company. Define the following predicates:

$A(x)$ :  $x$  is on the board of directors

$E(x)$ :  $x$  earns more than \$100,000

$W(x)$ :  $x$  works more than 60 hours per week

Translate the following logical expressions into English:



(a)  $\forall x (A(x) \rightarrow E(x))$

Solution: Every member of the board of directors earns more than \$100,000.

(b)  $\exists x (E(x) \wedge \neg W(x))$

Solution: There is someone who earns more than \$100,000, but does not work more than 60 hours per week.

(c)  $\forall x (W(x) \rightarrow E(x))$

Solution: Everyone who works more than 60 hours per week earns more than \$100,000.

(d)  $\exists x (\neg A(x) \wedge E(x))$

Solution: There is someone who is not on the board of directors and earns more than \$100,000.

(e)  $\forall x (E(x) \rightarrow (A(x) \vee W(x)))$

Solution: Everyone who earns more than \$100,000 is on the board of directors or works more than 60 hours per week.

(f)  $\exists x (A(x) \wedge \neg E(x) \wedge W(x))$

Solution: Someone on the board of directors does not earn more than \$100,000 and works more than 60 hours per week.

### Exercise 1.6.6

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

$P(x)$ :  $x$  was given the placebo

$D(x)$ :  $x$  was given the medication

$A(x)$ :  $x$  had fainting spells

$M(x)$ :  $x$  had migraines

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

$P(x)$	$D(x)$	$A(x)$	$M(x)$		
Frodo	T	F	F	T	
Gandalf		F	T	F	F
Gimli	F	T	T	F	
Aragorn		T	F	T	T
Bilbo	T	T	F	F	

For each of the following quantified statements, indicate whether the statement is a proposition. If the statement is a proposition, give its truth value and translate the expression into English.

(a)  $\exists x (M(x) \wedge D(x))$

Solution: Proposition. False. One of the patients took the medication and had migraines.

(b)  $\exists x M(x) \wedge \exists x D(x)$

Solution: Proposition. True. One of the patients took the medication and one of the patients had migraines.

(c)  $\exists x M(x) \wedge D(x)$

Solution: Not a proposition.

(d)  $\forall x (A(x) \vee M(x))$

Solution: Proposition. False. Every patient had fainting spells or migraines or both.

(e)  $\forall x (M(x) \leftrightarrow A(x))$

Solution: Proposition. False. Every patient who had migraines had fainting spells and vice versa.

(f)  $\forall x ((M(x) \wedge A(x)) \rightarrow \neg D(x))$

Solution: Proposition. True. Every patient who had migraines and fainting spells did not take the medication.

(g)  $\exists x (D(x) \wedge \neg A(x) \wedge \neg M(x))$

Solution: Proposition. True. One of the patients who took the medication had neither fainting spells nor migraines.

(h)  $\forall x (D(x) \rightarrow (A(x) \vee M(x)))$

Solution: Proposition. False. Every patient who took the medication had fainting spells or migraines.

## Section 1.7

### Exercise 1.7.1

Apply De Morgan's law to each expression to obtain an equivalent expression in which each negation sign applies directly to a predicate. (i.e.,  $\exists x (\neg P(x) \vee \neg Q(x))$  is an acceptable final answer, but not  $\neg \exists x P(x)$  or  $\exists x \neg(P(x) \wedge Q(x))$ ).

(a)  $\neg \exists x P(x)$

Solution:  $\forall x \neg P(x)$

(b)  $\neg \exists x (P(x) \vee Q(x))$

Solution:  $\forall x (\neg P(x) \wedge \neg Q(x))$

(c)  $\neg \forall x (P(x) \wedge Q(x))$

Solution:  $\exists x (\neg P(x) \vee \neg Q(x))$

(d)  $\neg \forall x (P(x) \wedge (Q(x) \vee R(x)))$

Solution:  $\exists x (\neg P(x) \vee (\neg Q(x) \wedge \neg R(x)))$

### Exercise 1.7.2

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

$P(x)$ :  $x$  was given the placebo

$D(x)$ :  $x$  was given the medication

$M(x)$ :  $x$  had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Some patient was given the placebo and the medication.

$\exists x (P(x) \wedge D(x))$

Negation:  $\neg \exists x (P(x) \wedge D(x))$

Applying De Morgan's law:  $\forall x (\neg P(x) \vee \neg D(x))$

English: Every patient was either not given the placebo or not given the medication (or both).

(a) Every patient was given the medication.

Solution

$\forall x D(x)$

Negation:  $\neg \forall x D(x)$

Applying De Morgan's law:  $\exists x \neg D(x)$

English: Some patient was not given the medication.

(b) Every patient was given the medication or the placebo or both.

Solution

$\forall x (D(x) \vee P(x))$

Negation:  $\neg \forall x (D(x) \vee P(x))$

Applying De Morgan's law:  $\exists x (\neg D(x) \wedge \neg P(x))$

English: There is a patient who was not given the medication and not given the placebo.

(c) There is a patient who took the medication and had migraines.

Solution

$\exists x (D(x) \wedge M(x))$

Negation:  $\neg \exists x (D(x) \wedge M(x))$

Applying De Morgan's law:  $\forall x (\neg D(x) \vee \neg M(x))$

English: Every patient did not get the medication or did not have migraines or both.

(d) Every patient who took the placebo had migraines. (Hint: you will need to apply the conditional identity,  $p \rightarrow q \equiv \neg p \vee q$ .)

Solution

$\forall x (P(x) \rightarrow M(x))$

Negation:  $\neg \forall x (P(x) \rightarrow M(x))$

Applying De Morgan's law:  $\exists x (P(x) \wedge \neg M(x))$

English: Some patient took the placebo and did not have migraines.

### Exercise 1.7.3

In the following question, the domain of discourse is a set of students who show up for a test. Define the following predicates:

$P(x)$ :  $x$  showed up with a pencil

$C(x)$ :  $x$  showed up with a calculator

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Every student showed up with a calculator.

$\forall x C(x)$

Negation:  $\neg \forall x C(x)$

Applying De Morgan's law:  $\exists x \neg C(x)$

English: Some student showed up without a calculator.

(a) One of the students showed up with a pencil.

Solution

$\exists x P(x)$

Negation:  $\neg \exists x P(x)$

Applying De Morgan's law:  $\forall x \neg P(x)$

English: Every student showed up without a pencil.

(b) Every student showed up with a pencil or a calculator (or both).

Solution

$\forall x (P(x) \vee C(x))$

Negation:  $\neg \forall x (P(x) \vee C(x))$

Applying De Morgan's law:  $\exists x (\neg P(x) \wedge \neg C(x))$

English: Some student showed up with no pencil and no calculator.

(c) Every student who showed up with a calculator also had a pencil.

Solution

$\forall x (C(x) \rightarrow P(x))$

Negation:  $\neg \forall x (C(x) \rightarrow P(x))$

Applying De Morgan's law:  $\exists x (C(x) \wedge \neg P(x))$

English: Some student showed up with a calculator and no pencil.

(d) There is a student who showed up with both a pencil and a calculator.

Solution

$\exists x (P(x) \wedge C(x))$

Negation:  $\neg \exists x (P(x) \wedge C(x))$

Applying De Morgan's law:  $\forall x (\neg P(x) \vee \neg C(x))$

English: Every student who showed up did not have a pencil or did not have a calculator (or both).

#### Exercise 1.7.4

Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a)  $\neg \forall x (P(x) \wedge \neg Q(x)) \equiv \exists x (\neg P(x) \vee Q(x))$

Solution

$\neg \forall x (P(x) \wedge \neg Q(x))$

$\exists x \neg (P(x) \wedge \neg Q(x))$  De Morgan's law

$\exists x (\neg P(x) \vee \neg\neg Q(x))$  De Morgan's law  
 $\exists x (\neg P(x) \vee Q(x))$  Double negation law

(b)  $\neg\forall x (\neg P(x) \rightarrow Q(x)) \equiv \exists x (\neg P(x) \wedge \neg Q(x))$

Solution

$\neg\forall x (\neg P(x) \rightarrow Q(x))$   
 $\exists x \neg(\neg P(x) \rightarrow Q(x))$  De Morgan's law  
 $\exists x \neg(\neg\neg P(x) \vee Q(x))$  Conditional identity  
 $\exists x \neg(P(x) \vee Q(x))$  Double negation law  
 $\exists x (\neg P(x) \wedge \neg Q(x))$  De Morgan's law

(c)  $\neg\exists x (\neg P(x) \vee (Q(x) \wedge \neg R(x))) \equiv \forall x (P(x) \wedge (\neg Q(x) \vee R(x)))$

Solution

$\neg\exists x (\neg P(x) \vee (Q(x) \wedge \neg R(x)))$   
 $\forall x \neg(\neg P(x) \vee (Q(x) \wedge \neg R(x)))$  De Morgan's law  
 $\forall x (\neg\neg P(x) \wedge \neg(Q(x) \wedge \neg R(x)))$  De Morgan's law  
 $\forall x (P(x) \wedge \neg(Q(x) \wedge \neg R(x)))$  Double negation law  
 $\forall x (P(x) \wedge (\neg Q(x) \vee \neg\neg R(x)))$  De Morgan's law  
 $\forall x (P(x) \wedge (\neg Q(x) \vee R(x)))$  Double negation law

## Section 1.8

### Exercise 1.8.1

The table below shows the value of a predicate  $M(x, y)$  for every possible combination of values of the variables  $x$  and  $y$ . The domain for  $x$  and  $y$  is  $\{1, 2, 3\}$ . The row number indicates the value for  $x$  and the column number indicates the value for  $y$ . For example  $M(1, 2) = F$  because the value in row 1, column 2, is F.

M	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Indicate whether each of the logical expressions is a proposition. If so, indicate whether the proposition is true or false.

(a)  $M(1, 1)$

Solution: Proposition. True.

(b)  $\forall y M(x, y)$

Solution: Not a proposition. The variable  $x$  is not bound.

(c)  $\exists x M(x, 3)$

Solution: Proposition. True: for  $x = 1$  or  $x = 2$ ,  $M(x, 3)$  is true.

(d)  $\exists x \exists y M(x, y)$

Solution: Proposition. True: for example, for  $x = 1$  and  $y = 3$ ,  $M(x, y)$  is true.

(e)  $\exists x \forall y M(x, y)$

Solution: Proposition. False: there is no  $x$  such that  $M(x, 1)$ ,  $M(x, 2)$  and  $M(x, 3)$  are all true.

(f)  $M(x, 2)$

Solution: Not a proposition. The variable  $x$  is not bound.

(g)  $\exists y \forall x M(x, y)$

Solution: Proposition. True. For  $y = 1$ ,  $M(1, y)$ ,  $M(2, y)$ , and  $M(3, y)$  are all true.

### Exercise 1.8.2

The tables below show the values of predicates  $P(x, y)$ ,  $Q(x, y)$ , and  $S(x, y)$  for every possible combination of values of the variables  $x$  and  $y$ . The domain for  $x$  and  $y$  is  $\{1, 2, 3\}$ .

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F
Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F
S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Indicate whether each of the quantified statements is true or false.

(a)  $\exists x \forall y P(x, y)$

Solution: False. There is no  $x$  such that  $P(x, 1)$ ,  $P(x, 2)$  and  $P(x, 3)$  are all true.

(b)  $\exists x \forall y Q(x, y)$

Solution: True. For  $x = 2$ ,  $Q(x, 1)$ ,  $Q(x, 2)$  and  $Q(x, 3)$  are all true.

(c)  $\exists x \forall y P(y, x)$

Solution: True. For  $x = 1$ ,  $P(1, x)$ ,  $P(2, x)$  and  $P(3, x)$  are all true.

(d)  $\exists x \exists y S(x, y)$

Solution: False.  $S(x, y)$  is not true for any values for  $x$  and  $y$ .

(e)  $\forall x \exists y Q(x, y)$

Solution: False. When  $x = 1$ , there is no  $y$  such that  $Q(x, y)$  is true.

(f)  $\forall x \exists y P(x, y)$

Solution: True. For every  $x$ ,  $y$  can be chosen so that  $P(x, y)$  is true.

(g)  $\forall x \forall y P(x, y)$

Solution: False. For example, when  $x = y = 3$ ,  $P(x, y)$  is false.

(h)  $\exists x \exists y Q(x, y)$

Solution: True. For example, when  $x = 3$  and  $y = 1$ ,  $Q(x, y)$  is true.

(i)  $\forall x \forall y \neg S(x, y)$

Solution: True. For every value for  $x$  and  $y$ ,  $\neg S(x, y)$  is true.

### Exercise 1.8.3

Determine the truth value of each expression below. The domain is the set of all real numbers.

(a)  $\forall x \exists y (xy > 0)$

Solution: False. If  $x = 0$ , there is no  $y$  such that  $xy > 0$ .

(b)  $\exists x \forall y (xy = 0)$

Solution: True. If  $x = 0$ , then for all  $y$ ,  $xy = 0$ .

(c)  $\forall x \forall y \exists z (z = (x - y)/3)$

Solution: True. For any  $x$  and  $y$ , set the value of  $z$  to be  $(x - y)/3$ .

(d)  $\forall x \exists y \forall z (z = (x - y)/3)$

Solution: False. Once  $x$  and  $y$  are determined, then if  $z$  is any value besides  $(x - y)/3$ , the equality is false.

(e)  $\forall x \forall y (xy = yx)$

Solution: True.

(f)  $\exists x \exists y \exists z (x^2 + y^2 = z^2)$

Solution: True. One example is  $x = 3$ ,  $y = 4$ ,  $z = 5$ .

(g)  $\forall x \exists y y^2 = x$

Solution: False. If  $x < 0$ , then there is no  $y$  such that  $y^2 = x$ .

(h)  $\forall x \exists y (x < 0 \vee y^2 = x)$

Solution: True. If  $x \geq 0$ , then there is a  $y$  such that  $y^2 = x$ .

(i)  $\exists x \exists y (x^2 = y^2 \wedge x \neq y)$

Solution: True. For example,  $x = 2$  and  $y = -2$ .

(j)  $\exists x \exists y (x^2 = y^2 \wedge |x| \neq |y|)$

Solution: False.

(k)  $\forall x \forall y (x^2 \neq y^2 \vee |x| = |y|)$



Solution: True. If the first part is false for some  $x$  and  $y$  (i.e.,  $x^2 \neq y^2$ ), then  $|x| = |y|$  is true.

#### Exercise 1.8.4

Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositional logic.

(a)  $\forall x \exists y \exists z P(y, x, z)$

Solution:  $\exists x \forall y \forall z \neg P(y, x, z)$

(b)  $\forall x \exists y (P(x, y) \wedge Q(x, y))$

Solution:  $\exists x \forall y (\neg P(x, y) \vee \neg Q(x, y))$

(c)  $\exists x \forall y (P(x, y) \rightarrow Q(x, y))$

Solution:  $\forall x \exists y (P(x, y) \wedge \neg Q(x, y))$

(d)  $\exists x \forall y (P(x, y) \leftrightarrow P(y, x))$

Solution:  $\forall x \exists y ((P(x, y) \wedge \neg P(y, x)) \vee (\neg P(x, y) \wedge P(y, x)))$

(e)  $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

Solution:  $\forall x \forall y \neg P(x, y) \vee \exists x \exists y \neg Q(x, y)$

#### Exercise 1.8.5

The domain for variables  $x$  and  $y$  is a group of people. The predicate  $F(x, y)$  is true if and only if  $x$  is a friend of  $y$ . For the purposes of this problem, assume that for any person  $x$  and person  $y$ , either  $x$  is a friend of  $y$  or  $x$  is an enemy of  $y$ . Therefore,  $\neg F(x, y)$  means that  $x$  is an enemy of  $y$ .

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

(a) Everyone is a friend of everyone.

Solution

$\forall x \forall y F(x, y)$

Negation:  $\neg \forall x \forall y F(x, y)$

Apply De Morgan's law:  $\exists x \exists y \neg F(x, y)$

English: Someone is an enemy of someone.

(b) Someone is a friend of someone.

Solution

$\exists x \exists y F(x, y)$

Negation:  $\neg \exists x \exists y F(x, y)$

Apply De Morgan's law:  $\forall x \forall y \neg F(x, y)$

English: Everyone is an enemy of everyone.

(c) Someone is a friend of everyone.

Solution

$\exists x \forall y F(x, y)$

Negation:  $\neg \exists x \forall y F(x, y)$

Apply De Morgan's law:  $\forall x \exists y \neg F(x, y)$

English: Everyone is an enemy of someone.

(d) Everyone is a friend of someone.

Solution

$\forall x \exists y F(x, y)$

Negation:  $\neg \forall x \exists y F(x, y)$

Apply De Morgan's law:  $\exists x \forall y \neg F(x, y)$

English: Someone is an enemy of everyone.

## Section 1.9

### Exercise 1.9.1

The domain of discourse for this problem is a group of three people who are working on a project. To make notation easier, the people are numbered 1, 2, 3. The predicate  $M(x, y)$  indicates whether  $x$  has sent an email to  $y$ , so  $M(2, 3)$  is read "Person 2 has sent an email to person 3." The table below shows the value of the predicate  $M(x, y)$  for each  $(x, y)$  pair. The truth value in row  $x$  and column  $y$  gives the truth value for  $M(x, y)$ .

M	1	2	3
1	T	T	T
2	T	F	T
3	T	T	F

Indicate whether the quantified statement is true or false. Justify your answer.

(a)  $\forall x \forall y M(x,y)$

Solution: False. Counterexamples:  $x = y = 2$  or  $x = y = 3$ .

(b)  $\forall x \forall y ((x \neq y) \rightarrow M(x,y))$

Solution: True. For every  $x$  and  $y$  such that  $x \neq y$ ,  $M(x, y)$  is true.

(c)  $\exists x \exists y \neg M(x,y)$

Solution: True. For example,  $x = y = 2$  or  $x = y = 3$ .

(d)  $\exists x \exists y ((x \neq y) \wedge \neg M(x,y))$

Solution: False. For every  $x$  and  $y$  such that  $x \neq y$ ,  $M(x, y)$  is true.

(e)  $\forall x \exists y \neg M(x,y)$

Solution: False. If  $x = 1$ , there is no  $y$  such that  $M(1, y)$  is false.

(f)  $\exists x \forall y M(x,y)$

Solution: True. If  $x = 1$ , then  $M(x, y)$  is true for all  $y$ .

### Exercise 1.9.2

Show that the two quantified statements in each problem are not logically equivalent by filling in a table so that, for the domain of discourse  $\{a, b, c\}$ , the values of the predicate  $P$  you select for the table causes one of the statements to be true and the other to be false. For example, the table below shows that  $\forall x \forall y P(x, y)$  and  $\exists x \exists y P(x, y)$  are not logically equivalent because for the given values of the predicate  $P$ ,  $\forall x \forall y P(x, y)$  is false and  $\exists x \exists y P(x, y)$  is true.

P	a	b	c
a	T	T	T
b	T	F	T
c	T	T	F

(a)  $\forall x \exists y P(x, y)$  and  $\exists x \forall y P(x, y)$

Solution

For the table below,  $\forall x \exists y P(x, y)$  is true and  $\exists x \forall y P(x, y)$  is false.

P	a	b	c
a	T	F	T
b	T	F	F
c	F	T	F

(b)  $\forall x \exists y ((x \neq y) \wedge P(x, y))$  and  $\forall x \exists y P(x, y)$

Solution

For the table below,  $\forall x \exists y P(x, y)$  is true and  $\forall x \exists y ((x \neq y) \wedge P(x, y))$  is false.

P	a	b	c
a	T	F	T
b	F	T	F
c	F	T	F

(c)  $\exists x \exists y (P(x, y) \wedge P(y, x))$  and  $\exists x \exists y P(x, y)$

Solution

For the table below,  $\exists x \exists y P(x, y)$  is true and  $\exists x \exists y (P(x, y) \wedge P(y, x))$  is false.

P	a	b	c
a	F	F	T
b	T	F	F
c	F	T	F

### Exercise 1.9.3

The domain of discourse is the members of a chess club. The predicate  $B(x, y)$  means that person  $x$  has beaten person  $y$  at some point in time. Give a logical expression equivalent to the following English statements. You can assume that it is possible for a person to beat himself or herself.

(a) Sam has been beaten by someone.

Solution:  $\exists x B(x, \text{Sam})$

(b) Everyone has been beaten before.

Solution:  $\forall x \exists y B(y, x)$  which is the same as  $\forall y \exists x B(x, y)$

(c) No one has ever beaten Nancy.

Solution:  $\neg \exists x B(x, \text{Nancy})$

(d) Everyone has won at least one game.

Solution:  $\forall x \exists y B(x, y)$

(e) No one has beaten both Ingrid and Dominic.

Solution:  $\neg \exists x (B(x, \text{Ingrid}) \wedge B(x, \text{Dominic}))$

(f) Josephine has beaten everyone else.

Solution:  $\forall x ((x \neq \text{Josephine}) \rightarrow B(\text{Josephine}, x))$

(g) Nancy has beaten exactly one person.

Solution:  $\exists x \forall y (B(\text{Nancy}, x) \wedge ((x \neq y) \rightarrow \neg B(\text{Nancy}, y)))$

(h) There are at least two members who have never been beaten.

Solution:  $\exists x \exists y ((x \neq y) \wedge \forall z (\neg B(z, x) \wedge \neg B(z, y)))$  or  $\exists x \exists y \forall z ((x \neq y) \wedge (\neg B(z, x) \wedge \neg B(z, y)))$

#### Exercise 1.9.4

The domain for the variables  $x$  and  $y$  are the set of musicians in an orchestra. The predicates  $S$ ,  $B$ , and  $P$  are defined as:

$S(x)$ :  $x$  plays a string instrument

$B(x)$ :  $x$  plays a brass instrument

$P(x, y)$ :  $x$  practices more than  $y$

Give a quantified expression that is equivalent to the following English statements:

(a) There are no brass players in the orchestra.

Solution:  $\neg \exists x B(x)$

(b) Someone in the orchestra plays a string instrument and a brass instrument.

Solution:  $\exists x (B(x) \wedge S(x))$

(c) There is a brass player who practices more than all the string players.

Solution:  $\exists x (B(x) \wedge \forall y ((S(y) \rightarrow P(x, y)))$  which is equivalent to  $\exists x \forall y (B(x) \wedge ((S(y) \rightarrow P(x, y)))$

(d) All the string players practice more than all the brass players.

Solution:  $\forall x \forall y ((S(x) \wedge B(y)) \rightarrow P(x, y))$

(e) Exactly one person practices more than Sam.

Solution:  $\exists x (P(x, \text{Sam}) \wedge \forall y ((x \neq y) \rightarrow \neg P(y, \text{Sam})))$  which is the same as  $\exists x \forall y (P(x, \text{Sam}) \wedge ((x \neq y) \rightarrow \neg P(y, \text{Sam})))$

(f) Sam practices more than anyone else in the orchestra.

Solution:  $\forall y ((y \neq \text{Sam}) \rightarrow P(\text{Sam}, y))$

#### Exercise 1.9.5

The domain for the first input variable to predicate  $T$  is a set of students at a university. The domain for the second input variable to predicate  $T$  is the set of Math classes offered at that university. The predicate  $T(x, y)$  indicates that student  $x$  has taken class  $y$ . Sam is a student at the university and Math 101 is one of the courses offered at the university. Give a logical expression for each sentence.

(a) Sam has taken Math 101.

Solution:  $T(\text{Sam}, \text{Math 101})$

(b) Every student has taken at least one math class.

Solution:  $\forall x \exists y T(x, y)$

(c) Every student has taken at least one class besides Math 101.

Solution:  $\forall x \exists y ((y \neq \text{Math } 101) \wedge T(x, y))$

(d) There is a student who has taken every math class besides Math 101.

Solution:  $\exists x \forall y ((y \neq \text{Math } 101) \rightarrow T(x, y))$

(e) Everyone besides Sam has taken at least two different math classes.

Solution:  $\forall x \exists y \exists z ((x \neq \text{Sam}) \rightarrow ((y \neq z) \wedge T(x, y) \wedge T(x, z)))$

(f) Sam has taken exactly two math classes.

Solution:  $\exists y \exists z \forall w ((z \neq y) \wedge T(\text{Sam}, y) \wedge T(\text{Sam}, z) \wedge ((w \neq y \wedge w \neq z) \rightarrow \neg T(\text{Sam}, w)))$

### Exercise 1.9.6

Translate each of the following English statements into logical expressions. The domain of discourse is the set of all real numbers.

(a) There are two numbers whose ratio is less than 1.

Solution:  $\exists x \exists y (x/y < 1)$

(b) The reciprocal of every positive number is also positive.

Solution:  $\forall x ((x > 0) \rightarrow (1/x > 0))$

(c) There are two numbers whose sum is equal to their product.

Solution:  $\exists x \exists y (x + y = xy)$

(d) The ratio of every two positive numbers is also positive.

Solution:  $\forall x \forall y (((x > 0) \wedge (y > 0)) \rightarrow (x/y > 0))$

(e) The reciprocal of every positive number less than one is greater than one.

Solution:  $\forall x (((x > 0) \wedge (x < 1)) \rightarrow (1/x > 1))$

(f) There is no smallest number.

Solution:  $\neg \exists x \forall y (x \leq y)$

(g) Every number besides 0 has a multiplicative inverse.

Solution:  $\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$  which is equivalent to  $\forall x \exists y ((x \neq 0) \rightarrow (xy = 1))$

(h) Every number besides 0 has a unique multiplicative inverse.

Solution:  $\forall x \exists y (((x \neq 0) \rightarrow (xy = 1)) \wedge \forall z ((z \neq y) \rightarrow (xz \neq 1)))$  which is equivalent to  $\forall x \exists y \forall z (((x \neq 0) \rightarrow (xy = 1)) \wedge ((z \neq y) \rightarrow (xz \neq 1)))$

## Section 1.10

### Exercise 1.10.1

Indicate whether the argument is valid or invalid. For valid arguments, prove that the argument is valid using a truth table. For invalid arguments give truth values for the variables showing that the argument is not valid.

(a)

$p \vee q$

$p$

$\therefore q$

Solution

Not valid.  $p = T$  and  $q = F$ .

(b)

$p \leftrightarrow q$

$p \vee q$

$\therefore p$

Solution

Valid. The only row in which the hypotheses  $p \leftrightarrow q$  and  $p \vee q$  are both true is the first row, and the conclusion  $p$  is true in the first row.

$p$	$q$	$p \leftrightarrow q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	T	F

(c)

$p$

$q$

$\therefore p \leftrightarrow q$

Solution

Valid. The only row in which the hypotheses  $p$  and  $q$  are both true is the first row, and the conclusion  $p \leftrightarrow q$  is true in the first row.

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(d)

$p \vee q$

$\neg q$

$\therefore p \leftrightarrow q$

Solution

Not valid.  $p = T$  and  $q = F$ .

(e)

$(p \wedge q) \rightarrow r$

$\therefore (p \vee q) \rightarrow r$

Solution

Not valid.  $r = F, p = T, q = F$ , or  $r = F, p = F, q = T$ .

(f)

$(p \vee q) \rightarrow r$

$\therefore (p \wedge q) \rightarrow r$

Solution

Valid. For every row in which  $(p \vee q) \rightarrow r$  is true (rows 1, 3, 5, 7, and 8),  $(p \wedge q) \rightarrow r$  is also true.

$p$	$q$	$r$	$(p \vee q) \rightarrow r$	$(p \wedge q) \rightarrow r$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

### Exercise 1.10.2

Which of the following arguments are invalid and which are valid? Prove your answer by replacing each proposition with a variable to obtain the form of the argument. Then prove that the form is valid or invalid.

(a)

The patient has high blood pressure or diabetes or both.

The patient has diabetes or high cholesterol or both.

$\therefore$  The patient has high blood pressure or high cholesterol.

Solution

Not valid. Assign:

b: the patient has high blood pressure



d: the patient has diabetes  
c: the patient has high cholesterol  
The form of the argument is

$b \vee d$   
 $d \vee c$   
 $\therefore b \vee c$

The argument is not valid because when  $d = T$  and  $b = c = F$ , both hypotheses are true but the conclusion is false.

(b)  
He studied for the test or he failed the test or both.  
He passed the test.  
 $\therefore$  He studied for the test.

Solution  
Valid. Assign:

s: he studied for his test  
f: he failed the test  
The form of the argument is

$s \vee f$   
 $\neg f$   
 $\therefore s$

The hypotheses  $s \vee f$  and  $\neg f$  are both true only in the second line of the truth table below. The conclusion  $s$  is true in the second row.

s	f	$s \vee f$	$\neg f$
T	T	T	F
T	F	T	T
F	T	T	F
F	F	F	T

(c)  
If  $2\sqrt{2}$  is an irrational number, then  $22\sqrt{2}$  is an irrational number.  
 $22\sqrt{2}$  is an irrational number.  
 $\therefore 2\sqrt{2}$  is an irrational number.

Solution  
Not valid. Assign:

a:  $2\sqrt{2}$  is an irrational number

b:  $22\sqrt{22}$  is an irrational number

The form of the argument is

$a \rightarrow b$

b

$\therefore a$

The argument is not valid because when  $a = F$  and  $b = T$ , both hypotheses are true but the conclusion is false.