Note: GCF = GCD (greatest common factor = greatest common divisor)

1. *Find* the greatest non-zero digit "d" for which the number represented by 7743657131d is divisible by:
   a. 2 b. 4 c. 8 d. 5 e. 9 f. 11 g. 99

2. Which of the following numbers are prime? Which are composite?
   a. 1 b. 31 c. 51 d. 87 e. 91 f. 317 g. 517

3. *Simplify* the following exponential expressions (to a^b form where possible):
   a. $5^2 \cdot 5^0 \cdot (5^3)^2$ b. $5^2 \cdot 5^4 \cdot 125^3$ c. $27 + 6 \cdot 3^2$

4. *Write the prime factorization* in exponential form for a. 101898 b. 12857 c. 361323

5. *Factor* 420, 441, 462 as the products of primes.

6. *Find* the greatest common factor (GCF-GCD) and least common multiple (LCM) of 420, 441, and 462.

7. *Find* the GCF and LCM of $2^5 \cdot 5^2 \cdot 7 \cdot 17$ and $2 \cdot 3^5 \cdot 7^3 \cdot 11$, using the most efficient method.

8. *Find* the GCF and LCM of 41535 and 20709. Use the most efficient method.

9. Create and describe a "divisibility test" for divisibility by 45.

10. A train to Newark leaves the station every 48 minutes; a train to Brunswick leaves the station every 54 minutes. If both trains left the station at 10 AM, *at what time* will the trains next leave simultaneously?

11. Which of the following pairs of numbers are relatively prime?
    a. (123,456) b. (33,4576) c. (72,55) d. (29,108) e. (3611136,109005) f. (25, 9472) g. (1,6) h. (2,9)

12. In each of the following, what whole number(s) can replace X and make the statement true?
    a. $2^3 \cdot 3 \cdot 5^2 \cdot X = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$ b. $X \mid 30$ and $X \mid 147$ c. $X^2 \mid 98$ d. $6 \mid X$ and $10 \mid X$

13. Use an *example to illustrate* or a *counterexample to disprove* each statement: *(a,b,c are assumed to be in Z)*
    a. If $a \mid b$ then LCM(a,b) = b. b. If a is an even whole number, then $0 \mid a$.
    c. GCF(a,b) $\mid$ LCM(a,b). d. LCM(a,b) $\mid$ GCF(a,b).
    e. If $a \nmid b$ and $a \nmid c$, then $a \nmid (b+c)$ f. If $a \nmid b$ then a & b are relatively prime.
    g. If $a \mid b$, then LCM(a,b) = a·b. h. If $3 \mid a$ and $6 \mid a$ then $18 \mid a$.
    I. If $a \mid b$, then GCF(a,b) = b. k. If $a \mid b$ & $b \mid c$, then $a \mid c$.
    j. If the sum of the digits of a whole number is divisible by 6, then the number is divisible by 6.

14. a. *How many divisors* has 32? 360? b. Find them. c. *What is the smallest* number with exactly eleven divisors? d. What is the smallest number > 10 that, when divided by 15, 20 and 25, leaves remainder 3?

15. Is $29 \cdot 2^{17} \cdot 3^9 \cdot 7^{22} + 52$ divisible by 29? *How do you know?*

16. What is the greatest *prime* that requires checking to test whether 1217 is prime? *Why* (briefly)?

17. If New Year's Day falls on a Sunday, & the year is odd, *on what day of the week* will next New Year's Day fall?

18. A proper divisor (*pd*) of a number is a positive divisor less than the number.
    a. *Find* the *pd*'s of twelve. b. *Prove or disprove* that every composite has at least three *pd*'s.

19. A lot of candy bars sold for $39.59 in one hour after the price was reduced from the original 49 cents per bar. How many candy bars were sold? *How do you know?*

20. A nursery has 144 tuberoses, 252 oriental lilies and 540 dahlias. They want to put together mini-collections *each* containing the *same numbers* of bulbs.
    a. What is the maximum number of collections using all the bulbs?
    b. How many of each type of flower bulb will be in each package?

21. Given LCM(15, b) = 180 and GCF (15, b) = 3, what is b?

22. **Given*: LCM(a,b) = 240 and GCF(a.b) = 12. what are the possibilities for a and b?

23. *Would our test for divisibility by 3 work in other bases, e.g. base 5? Why or why not?

24. **(a real brain teaser) *Find* the *remainder* that results when $10^{101} \cdot 12^{88}$ is divided by 11.

1. a. 8 b. 6 c. 2 d. 5 e. 1 f. 1 g. 1 (makes number divisible by both 9 & 11, thus by 99)

2. Prime: (b,f) Composite (c,d,e,g) *(...and what about a? 1 is neither prime nor composite!)*

3. a. 5^8 b. 5^{15} c. $3^3 + 2\cdot3^3 = (1+2)\cdot3^3 = 3\cdot3^3 = 3^4$

4. a. $101898 = 2\cdot3^4\cdot17\cdot37$ b. $12857 = 13\cdot23\cdot43$ c. $361323 = 3^2\cdot19\cdot2113$

5. $420 = 4\cdot105 = 2^2\cdot3\cdot5\cdot7$
 $441 = 9\cdot49 = 3^2\cdot7^2$
 $462 = 2\cdot231 = 2\cdot3\cdot7\cdot11$

6. GCF(420,441,462) = $3\cdot7 = 21$
 LCM(420,441,462) = $2^2\cdot3^2\cdot5\cdot7^2\cdot11 = 97020$

7. GCF($2^5\cdot5^2\cdot7\cdot17$, $2\cdot3^5\cdot7^3\cdot11$) = 14
 LCM = $2^5\cdot3^5\cdot5^2\cdot7^3\cdot11\cdot17 = 12469010400$

8. $41535 = \underline{2}\cdot20709 + \underline{117}$ → 117 is the GCF(41535, 20709)
 $20709 = \underline{177}\cdot117 + \underline{0}$ LCM(41535, 20709) = (41535·20709)/117 = 41535·177 = 7351695 *(see #21)*

9. Does the number end in 0 or 5, and do its digits add to a multiple of 9? *If yes to both, then the number is a multiple of both 5 and 9, therefore a multiple of 45; if either part fails, number is not multiple of 45.*

10. LCM(48,54) = 432. Next simultaneous departure will be at 17:32 (*432 min = 7hr,32min after 10 AM*).

11. cdfg & h. (b: 33= 3·11 & 11|4576 a,e: 3|both numbers. f: 5 is the only prime factor of 25, & 5∤ .)

12. a. 2·3·7 b. 1, 3 c. 1, 7 (*as* $98 = 2\cdot7^2$) d. any multiple of 30, *the LCM(6,10)*

13. a. True: *for example:* let a = 6, b = 18. 6|18. LCM(a,b) = 18 = b
 b. False: let a = 4. 4 isn't a multiple of 0; *0 · c = 4* has no solution, because 0 · c = 0, never anything else.
 c. True: *for example:* let a = 6, b = 10. GCF(6,10) = 2. LCM(6,10) = 30. 2|30.
 d. False: LCM(6,10) = 30, GCF(6,10) = 2 and 30 does not divide 2 !!! *(GCF is small ! LCM is large !!)*
 e. False: consider 2, and 3+ 5.
 f. False: let a = 9, b = 12. 9∤12. 3|12 and 3|9, so 9 and 12 are not relatively prime.
 g. False: see part a.
 h. False: let a = 6. Then 3|a and 6|a , but 18 does not. *(The problem is: 3 & 6 are **not** relatively prime.)*
 i. False: let a = 3, b = 12 GCF(9,12) = 3, not 12. *(If a| b, then GCF(a,b) is a ...not b.)*
 j. True: 3|27 & 27|270. Therefore 3|270. *(This is the transitive property of the | relation.)*
 k. False: consider 15, or 33, or ...

14. $32 = 2^5$ has 6 divisors: 1, 2, 4, 8, 16 & 32.
$360 = 2^3 3^2 5$ has 4·3·2 = 24 divisors: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360
($2^{0 \text{ or } 1 \text{ or } 2 \text{ or } 3}\, 3^{0 \text{ or } 1 \text{ or } 2}\, 5^{0 \text{ or } 1}$ → there are 4 x 3 x 2 possibilities).
Any number with exactly eleven divisors must be of the form p^{10}; the smallest such is 2^{10}. 14d. 303

15. Think of $29\cdot2^{17}\cdot3^9\cdot7^{22} + 52$ as a+ b : since 29|a, and 29∤b, it follows that 29∤a+ b.

16. *We notice that:* $35^2 = 1225 > 1217$ ... so the greatest *prime* we must check is *31*.
 (& this because the product of two primes> 31 will exceed 1217. If 1217 has factors, one must be ≤ 31.)

17. *("Year odd" tells us it's not a leap year.)* 365 = 52·7 + 1, so the next New Year's Day will fall on Monday

18. a. *pd's of 12:* 1,2,3,4,6 b. The square of any prime *p* (e.g. $p^2 = 9$) has only two proper divisors, *1* & *p*.

19. 3959¢ must be a multiple of X¢, and X < 49.
 2,3,5,7,11,13,17,19,23,29,31,41 43 and 47 do not divide 3959. 37 does.
 37 x 107 = 3959¢ as required, and is the only non-trivial factorization. X is 37(¢); there were 107 candy bars.

20. a. Number of packages *must divide* each of 144, 252, & 540. GCF($2^4 3^2$, $2^2 3^2 7$, $2^2 3^3 5$) = 36
 b. Each package will contain 144/36 = 4 tuberoses; 252/36 = 7 lilies; and 540/36 = 15 dahlias.

21. *Since LCM = a·b/GCF, it follows that:* b= LCM·GCF/a. *(You can also play around & get this.)* b = 36.

22. Since $\text{LCM}(a,b) = \dfrac{a\cdot b}{\text{GCD}(a,b)}$ we know 240 = a·b/12 & further, a= 12m & b = 12n for some m & n in *Z*. So 240 = 12m·12n/12 , which tells us 240 = 12m·n or m·n = 20 = $2^2\cdot5$.
 Since m & n must be relatively prime, m must be 5 and n must be 2^2... a = 12·5 & b = $12\cdot2^2$ *(or vice versa).*

23. Our test for div. by 3 works because our base, 10, is 9+ 1. Thus we should not expect it to work in other bases (that are not 1 more than a multiple of 3). E.g. $11_{\text{FIVE}} = 6_{\text{TEN}}$. 1+ 1= 2, is not divisible by 3, but 11_{FIVE} is.

24. $10^{101}\cdot12^{88} = (11-1)^{101}(11+1)^{88} = (11M + (-1)^{101})\cdot(11N + 1) = 11K - 1$, so remainder = 10 (−1...).

Practice Final Questions on algebra

1. Margaret writes the following equations in order to solve 87 – 15: 87 – 10 = 77 – 5 = 72.
   This is incorrect because, for example, 87 – 10 does not equal 77 – 5.
   Use Margaret's strategy, but write correct equations showing the steps
.

2. If n is a number, then "the sum of twice that number and 5" can be expressed as
   a. 2(n + 5) b. 2n - 5 c. 2n + 5 d. $n^2 + 5$

3. I am thinking of a number. If I add 7, then multiply the sum by 3 and then subtract 12, the result is 36. Use algebra to find the number.

4. Write an algebraic expression for the following:
   a. the number of inches in m yards
   b. the cost in dollars of a gym membership for t months if it costs $125 just to join (membership fee) and then $75 per month for each month the gym is used
   c. the perimeter of a rectangle with length k cm and width s cm

5. Illustrate the identity, $(a + b)(a + b) = a^2 + 2ab + b^2$, by a rectangular array.

6. Let a and b be rational numbers. Fill in the blanks to justify each step.

$(a + b)^2 = (a + b)(a + b)$ def'n of exponent 2
$= (a + b)a + (a + b)b$ ________________
$= (a^2 + ba) + (ab + b^2)$ ________________
$= a^2 + ba + ab + b^2$ ________________
$= a^2 + ab + ab + b^2$ ________________
$= a^2 + (ab + ab) + b^2$ ________________
$= a^2 + (1ab + 1ab) + b^2$ ________________
$= a^2 + (1+1)ab + b^2$ ________________
$= a^2 + 2ab + b^2$ 1+ 1 = 2

7. Factor the algebraic expression, $a^2 – b^2$. Use your result to calculate $32^2 – 28^2$ with only one multiplication. Show your steps.
   $a^2 – b^2 =$ ______________ $32^2 – 28^2 =$ ________________________________

10. Express $\frac{10^{15} \cdot 15^{23}}{30^7 \cdot 9^8}$ as a product of prime numbers.

11. Give teacher's solutions using algebra.
   a. The lengths of the sides of a triangle, measured in inches, are consecutive whole numbers. If the perimeter is 45 inches, what is the length of the shortest side?
   b. Sofia had five times as much money as Diego. After Sofia spent $62 and Diego received his $22 allowance, they had the same amount of money. How much did they each have at the beginning?

12. Give two teacher's solutions to each of the following problems,
   one using bar diagrams and the other using algebra.

a. There are three times as many boys as girls. If there are 96 children, how many girls are there?

b. There are three children in a family. Ed is 20 pounds heavier than Fred who weighs twice as much as Ned. If the three children weigh 180 pounds altogether, how much does Ed weigh?

c. John and Wendy have a total of 1012 pennies. Wendy has 134 less than John. How many has John?

d. A bag of cookies contains two varieties, chocolate chip and oatmeal. There are four times as many chocolate chip as oatmeal cookies. If there are 36 more chocolate chip than oatmeal cookies, how many cookies are there altogether?

Answers to Practice Final Questions on Algebra

1. Margaret writes the following equations in order to solve 87 – 15: 87 – 10 = 77 – 5 = 72.
This is incorrect because, for example, 87 – 10 does not equal 77 – 5.
Use Margaret's strategy, but write correct equations showing the steps.

87 – 15 = (87 – 10) – 5 = 77 – 5 = 72
as in "first subtract 10, then 5"

2. If n is a number, then "the sum of twice that number and 5" can be expressed as © 2n+5.
a. 2(n + 5) b. 2n - 5 c. 2n + 5 d. $n^2 + 5$

3. I am thinking of a number. If I add 7, then multiply the sum by 3 and then subtract 12, the result is 36.
Use algebra to find the number.

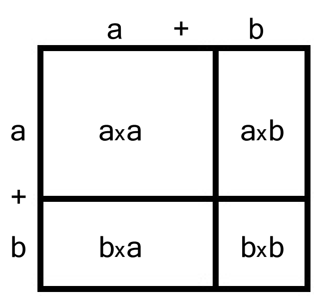
$(n+7)\cdot 3 - 12 = 36$
$3n + 21 - 12 = 36$
$3n + 9 = 36$
$3n = 27$ So n = 3

4. Write an algebraic expression for the following:
a. the number of inches in m yards
In 1 yard, there are 36 inches
In 2 yds, there are 72 inches
In m yds, there are 36·m inches.

b. the cost in dollars of a gym membership for t months if it costs $125 just to join (membership fee) and then $75 per month for each month the gym is used
one month's use of the gym costs initial fee plus monthly fee = $125 + $75
two month's use costs $125 + 2($75)
t month's use costs $125 + t($75)

c. the perimeter of a rectangle with length k cm and width s cm
Perimeter = distance around = (k cm + s cm) · 2

5. Illustrate the identity, $(a + b)(a + b) = a^2 + 2ab + b^2$, by a rectangular array.



$(a + b)^2$

IS

$a^2 + 2ab + b^2$

6. Let a and b be rational numbers. Fill in the blanks to justify each step.

$(a + b)^2$	$= (a + b)(a + b)$	def'n of exponent 2 (Definition of exponents. $w^2 = w\,w$)
	$= (a + b)a + (a + b)b$	Distributive property (of x over +)
	$= (a^2 + ba) + (ab + b^2)$	Distributive property (of x over +)
	$= a^2 + ba + ab + b^2$	Associative property of + (so no parentheses needed)
	$= a^2 + ab + ab + b^2$	Commutative property of multiplication
	$= a^2 + (ab + ab) + b^2$	Associative property of +
	$= a^2 + (1ab + 1ab) + b^2$	Identity property of mult.
	$= a^2 + (1+1)ab + b^2$	Identity property of mult.
	$= a^2 + 2ab + b^2$	1+ 1 = 2

7. Factor the algebraic expression, $a^2 – b^2$. Use your result to calculate $32^2 – 28^2$ with only one multiplication. Show your steps.

$a^2 - b^2 = (a + b) (a - b)$

$32^2 - 28^2 = (32 + 28)(32 - 28) = 60 \cdot 4 = 240$

10. Express $\frac{10^{15} \cdot 15^{23}}{30^{7} \cdot 9^{8}}$ as a product of prime numbers.

$$\frac{10^{15} \cdot 15^{23}}{30^{7} \cdot 9^{8}} = \frac{(2 \cdot 5)^{15} \cdot (3 \cdot 5)^{23}}{(2 \cdot 3 \cdot 5)^{7} \cdot (3 \cdot 3)^{8}} = \frac{2^{15} \cdot 5^{15} \cdot 3^{23} \cdot 5^{23}}{2^{7} \cdot 3^{7} \cdot 5^{7} \cdot 3^{16}} = \frac{2^{15}}{2^{7}} \cdot \frac{3^{23}}{3^{23}} \cdot \frac{5^{38}}{5^{7}} = 2^{8} \cdot 5^{31}$$

11. Give teacher's solutions using algebra.

a. The lengths of the sides of a triangle, measured in inches, are consecutive whole numbers. If the perimeter is 45 inches, what is the length of the shortest side?

Let the length of the shortest side be called x.
Then the other two are x+1 and x+2, because the three lengths are consecutive whole numbers.
The perimeter is the sum of the lengths of the sides, x + (x+1) + (x+2), and this is given to be 45 inches. So:

$$x + x+1 + x+2 = 45$$
$$3x + 3 = 45$$
$$3x = 42$$
$$x = 14$$

The length of the shortest side is 14 inches.
Check: The other two sides are thus 15 in and 16 in. 14 + 15 + 16 = 45.

b. Sofia had five times as much money as Diego. After Sofia spent \$62 and Diego received his \$22 allowance, they had the same amount of money. How much did they each have at the beginning?

Let x = the amount of money Diego had at the start of the description.
Then 5x = the amount Sofia had at the start.
After Sofia spends \$62 she will have 5x – 62.
Diego has x; after he receives \$22 allowance he has x + 22.
} These amounts are given to be equal.

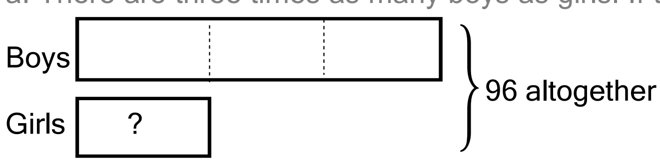
$$5x - 62 = x + 22$$
$$4x - 62 = 22$$
$$4x = 84$$
$$x = 21$$

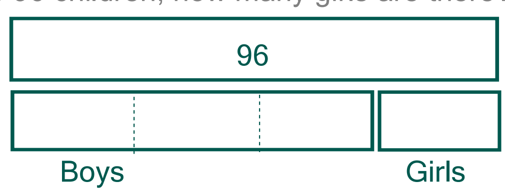
Diego had \$21.00 to begin.

Check: Diego had \$21 . Sofia had 5·\$21 = \$105
After changes, Diego will have 21+22 , Sofia will have \$43.

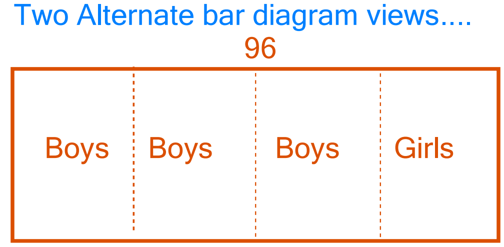
12. Give two teacher's solutions to each of the following problems, one using bar diagrams and the other using algebra.

a. There are three times as many boys as girls. If there are 96 children, how many girls are there?





Two Alternate bar diagram views....



4 x ? = 96
? = 96 ÷ 4 = 24

There are 24 girls.

Check: If 24 girls, then 3·24 = 72 boys.
24 + 72 = 96 ✔

With algebra:

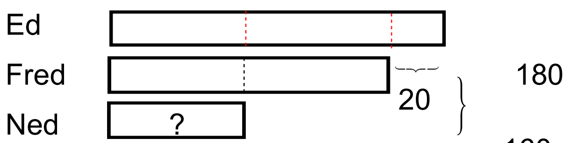
Let the number of girls = x.
Then number of boys = 3x
Since the total is 96, 3x + x = 96.

$$3x + x = 96$$
$$4x = 96$$
$$x = 24$$

There are 24 girls.

Check: 24 girls ⇒ 3·24 = 72 boys ⇒ 24 + 72 = 96 altogether. ✔

b. There are three children in a family. Ed is 20 pounds heavier than Fred who weighs twice as much as Ned. If the three children weigh 180 pounds altogether, how much does Ed weigh?



$$\begin{array}{r} 180 \\ -\ 20 \\ \hline 160 \end{array}$$

160 ← 3 boys' total weight without Ed's "extra" 20 pounds

160 ÷ 5 = 32.

Ned weighs 32 pounds so Fred weighs 64, and Ed, 84, pounds

Check: Ned 32, Fred 64, Ed 84. 32+64+84 = 180 ✔
x2 +20 ✔

With algebra:

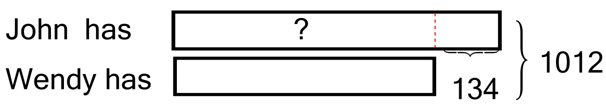
Let Ned's weight be x
Then Fred's weight must be 2x
and Ed's weight must be 2x + 20
The total of their weights is 180 pounds, so

$$\begin{aligned} x + 2x + 2x + 20 &= 160 \\ 5x + 20 &= 180 \\ 5x &= 160 \\ x &= 32 \end{aligned}$$

Ned weighs 32 pounds.
So Fred weighs 64 pounds, and Ed weighs 84 lbs.

Check is same as above.

c. John & Wendy have 1012 pennies. Wendy has 134 less than John. How many pennies has John?



$$\begin{array}{r} 1012 \\ -\ 134 \\ \hline 878 \end{array}$$

878 total pennies without John's "extra" 134.

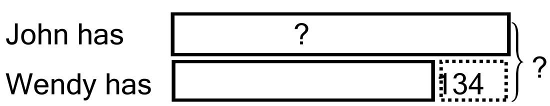
878 ÷ 2 = 439 Wendy has 439 pennies.

$$\begin{array}{r} 439 \\ +\ 134 \\ \hline 573 \end{array}$$

John has 573 pennies.

Check:
573+ 439 = 1012,
573—439 = 134 ✔

Alternate view:



$$\begin{array}{r} 1012 \\ +\ 134 \\ \hline 1146 \end{array}$$

If Wendy were to get 134 more pennies, then the total would be 134 more, and she and John would have equal amounts. This would be 1146 ÷ 2 = 573. John must have 573 pennies.

With algebra:

Let x = number of pennies Wendy has.
Then John has x+134 pennies.
The total number of pennies is 1012, so

$$\begin{aligned} x + x + 134 &= 1012 \\ 2x + 134 &= 1012 \\ 2x &= 878 \\ x &= 439 \end{aligned}$$

So Wendy has 439 pennies. Therefore John has 439 + 134 = 573

Alternate view:

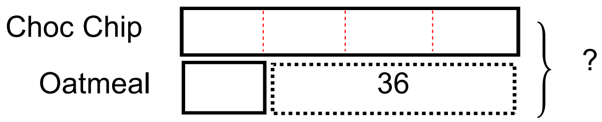
Let x = number of pennies John has
Then number of pennies Wendy has is x – 134
The total number of pennies is 1012, so

$$\begin{aligned} x + x - 134 &= 1012 \\ 2x - 134 &= 1012 \\ 2x &= 1146 \\ x &= 573 \end{aligned}$$

John has 573 pennies.

Check: 573 – 134 = 439. 573 + 439 = 1012. ✔

d. There are 4 times as many CC as OM cookies. 36 more CC than OM cookies, how many cookies total?



There are 60 cookies in total.
(12 are OM, and 48 are CC)

3 units = 36 cookies   36 ÷ 3 = 12
1 unit = 12 cookies

$$\begin{array}{r} 12 \\ \underline{\times\ 5} \\ 60 \end{array}$$

5 units = 60 cookies

Check: 48 ÷ 12 = 4, 48 — 12 = 36, 48+12 = 60 ✔

With algebra:

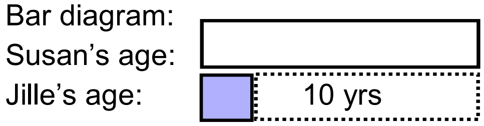
Let the number of OM (oatmeal) cookies be x.
Then the number of CC cookies must be 4x.
There are 36 more CC than OM. So

$$4x - x = 36$$
$$3x = 36$$
$$x = 12$$

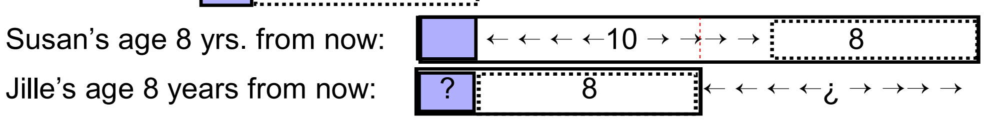
There are 12 OM cookies, and 48 CC cookies, and the total is 12 + 48 = 60 cookies.

For those who want a little more challenge,
here is one more word problem, another one of those "age" problems:

Susan is 10 years older than Jille. In 8 years she will be twice as old as Jille.
How old is Susan now?
Bar diagram:



Aaargh!!! (What to do with the rest of the info! Starting over....)



Double Aaargh!!!

Conclusion: This problem can be done, but is quite confusing without algebra. Here we go:
Once we have the diagram, we can see the difference, indicated by "¿", must be 18 – 8 = 10.
Since Susan's age at that point will be double Jille's, from the diagram we see that ? + 8 must also be 10.
Therefore, Jille's age now (?) must be ___ – ___ = ___.
And Susan is 10 years older than Jille, so Susan must be ___.
Check: eight years from now, Jille will be _____, and Susan will be _____.
Maybe you can find a better way to illustrate this problem!

With algebra:

Let x = Jille's age now.
Then Susan's age now is x + 10.
In 8 years, Jille's age will be x + 8,
and Susan's age will be (x + 10) + 8.
We know that Susan's age will be twice Jille's then:

$$(x + 10) + 8 = 2(x + 8)$$
$$x + 18 = 2x + 16$$
$$2 = x$$

So Jille is now 2 years old, and Susan is 2+ 10 = 12 years old.
Eight years from now, Jille will be ___ years old, and Susan will be ___ years old.
Will Susan be twice as old as Jille then?