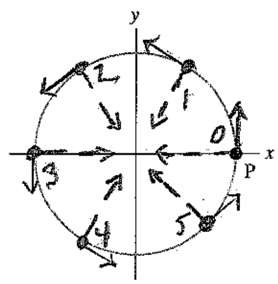
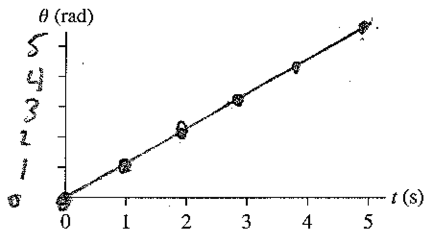
# 7 Rotational Motion

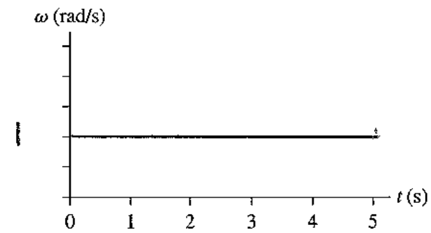
black ———
red - - -

## 7.1 Describing Circular and Rotational Motion

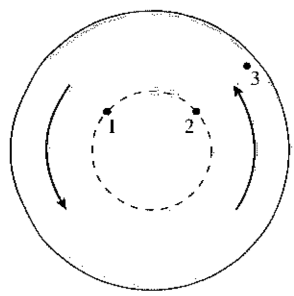
1. A particle undergoes uniform circular motion with constant angular velocity $\omega = +1.0$ rad/s, starting from point P.
   a. On the figure, draw a motion diagram showing the location of the particle every 1.0 s until the particle has moved through an angle of 5 rad. Draw velocity vectors **black** and acceleration vectors **red**. For this question, you can use 1 rad $\approx$ 60°.
   b. Below, graph the particle's angular position $\theta$ and angular velocity $\omega$ for the first 5 s of motion. Include an appropriate vertical scale on both graphs.




2. The figure shows three points on a steadily rotating wheel.
   a. Rank in order, from largest to smallest, the angular velocities $\omega_1$, $\omega_2$, and $\omega_3$ of these points.




   Order: $\omega_1 = \omega_2 = \omega_3$

   Explanation: Angular rotation is the same for all points on a rotating object.

   b. Rank in order, from largest to smallest, the speeds $v_1$, $v_2$, and $v_3$ of these points.

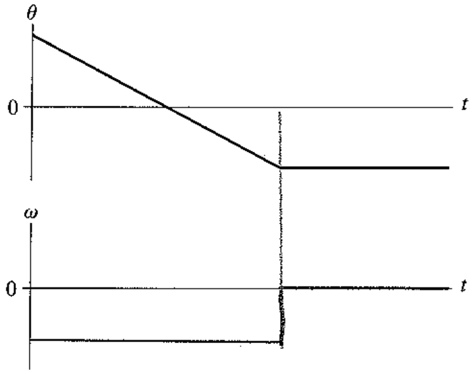
   Order: $v_3 > v_1 = v_2$

   Explanation:

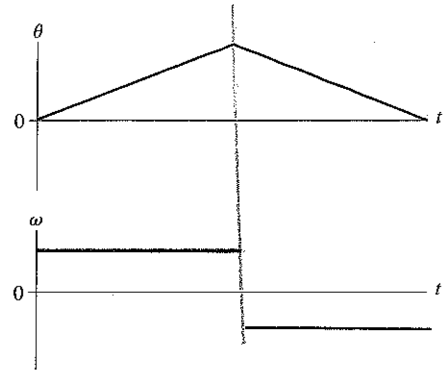
   $v = \omega r$, and $r_3 > r_1 = r_2$

3. Below are two angular position-versus-time graphs. For each, draw the corresponding angular velocity-versus-time graph directly below it.

a.

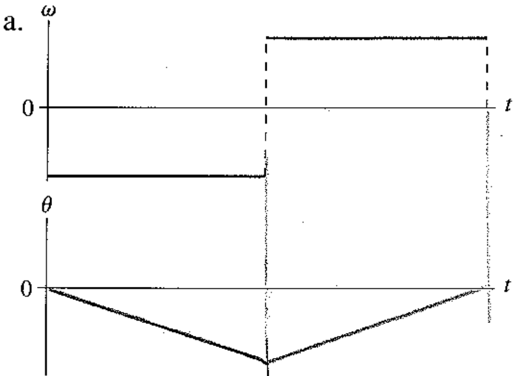



b.

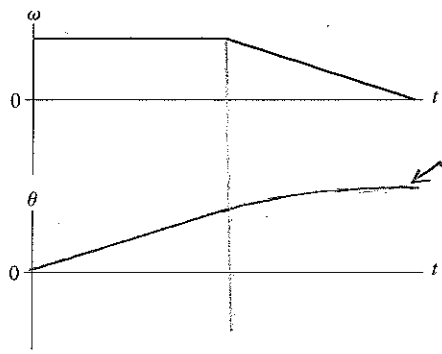



4. Below are two angular velocity-versus-time graphs. For each, draw the corresponding angular position-versus-time graph directly below it. Assume $\theta_0 = 0$ rad.

a.

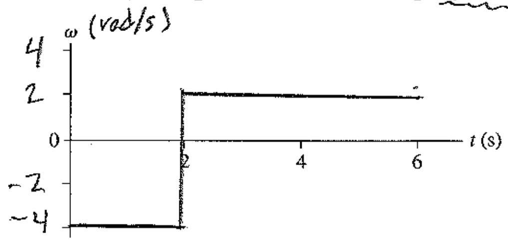


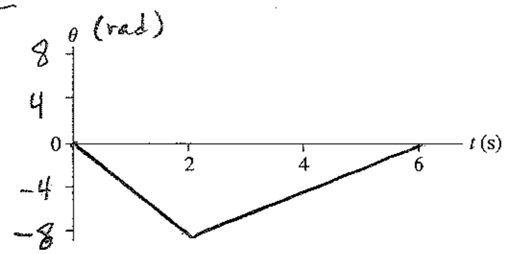

b.




zero slope here

5. A particle in circular motion rotates clockwise at 4 rad/s for 2 s, and then counterclockwise at 2 rad/s for 4 s. The time required to change direction is negligible. Graph the angular velocity and the angular position, assuming $\theta_0 = 0$ rad.




6. A particle moves in uniform circular motion with $a = 8\ \text{m/s}^2$. What is $a$ if

a. The radius is doubled without changing the angular velocity? $16\ \text{m/s}^2$

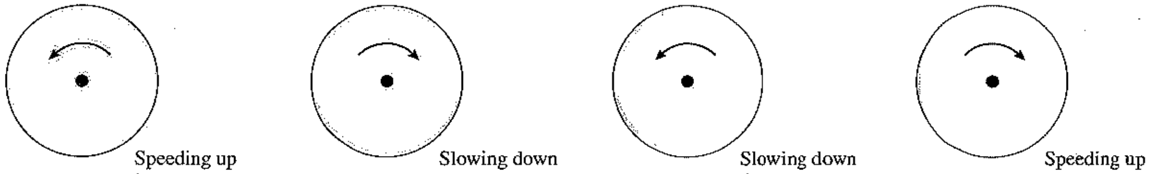
b. The radius is doubled without changing the particle's speed? $4\ \text{m/s}^2$

c. The angular velocity is doubled without changing the circle's radius? $32\ \text{m/s}^2$

$$a = \frac{v^2}{r} = \omega^2 r$$

## 7.2 The Rotation of a Rigid Body

7. The following figures show a rotating wheel. If we consider a counterclockwise rotation as positive (+) and a clockwise rotation as negative (−), determine the signs (+ or −) of $\omega$ and $\alpha$.




| | Speeding up | Slowing down | Slowing down | Speeding up |
|---|---|---|---|---|
| $\omega$ | + | − | + | − |
| $\alpha$ | + | + | − | − |

8. A ball is rolling back and forth inside a bowl. The figure shows the ball at extreme left edge of the ball's motion as it changes direction.

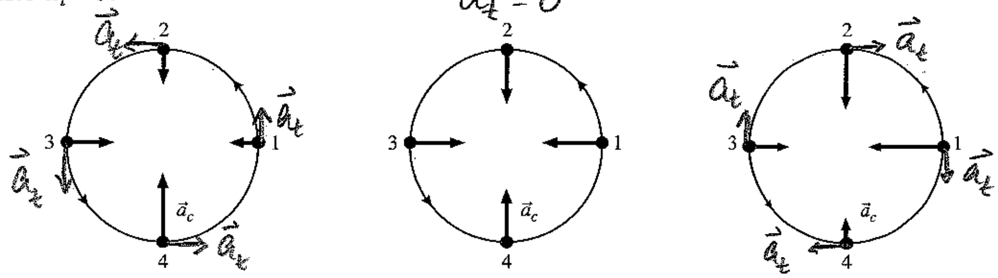
a. At this point, is $\omega$ positive, negative, or zero? zero

b. At this point, is $\alpha$ positive, negative, or zero? +

9. The figures below show the centripetal acceleration vector $\vec{a}_c$ at four successive points on the trajectory of a particle moving in a counterclockwise circle.

a. For each, draw the tangential acceleration vector $\vec{a}_t$ at points 2 and 3 or, if appropriate, write $\vec{a}_t = \vec{0}$.

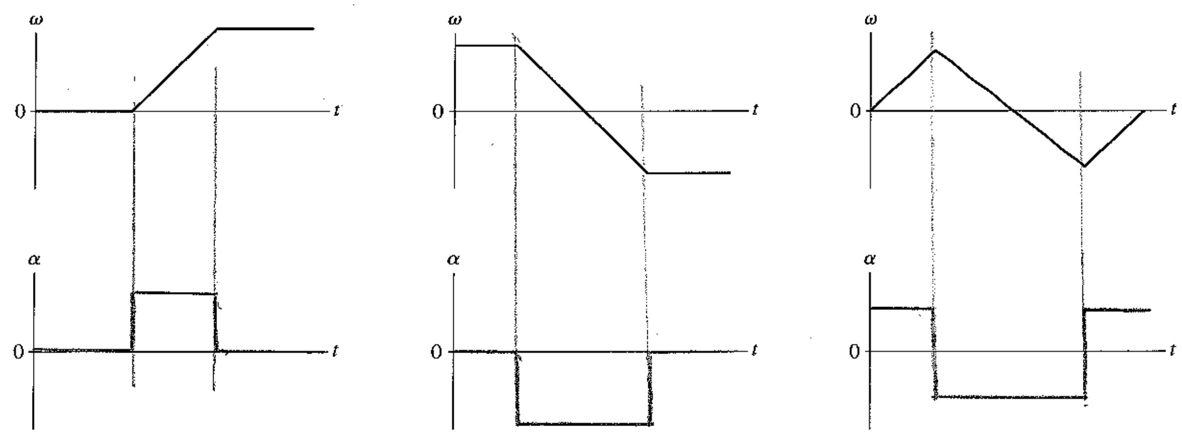
$\vec{a}_t = \vec{0}$



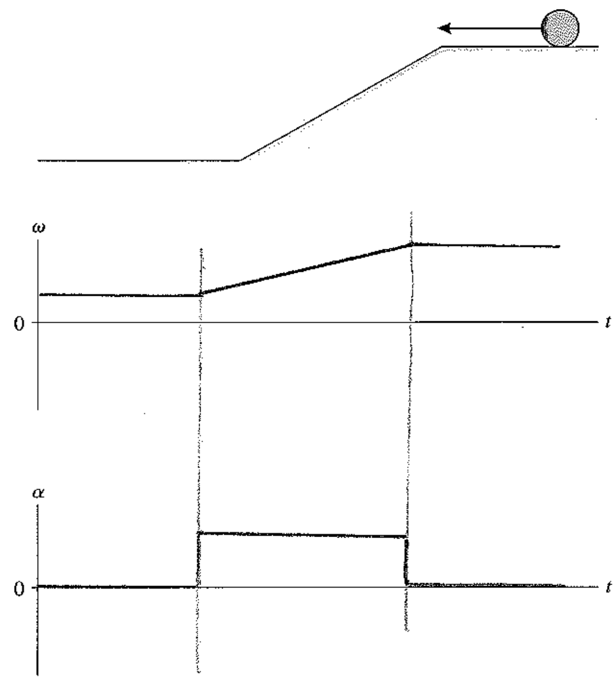

b. If we consider a counterclockwise rotation as positive and a clockwise rotation as negative, determine if the particle's angular acceleration $\alpha$ is positive (+), negative (−), or zero (0).

$\alpha =$ +　　$\alpha =$ 0　　$\alpha =$ −

10. Below are three angular velocity-versus-time graphs. For each, draw the corresponding angular acceleration-versus-time graph.

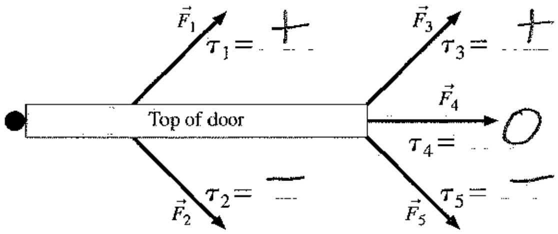



11. A wheel rolls to the left along a horizontal surface, down a ramp, and then continues along the lower horizontal surface. Draw graphs for the wheel's angular velocity $\omega$ and angular acceleration $\alpha$ as functions of time.

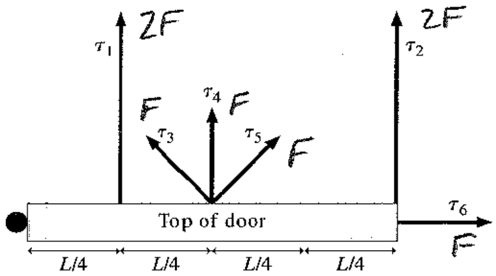


## 7.3 Torque

12. Five forces are applied to a door. For each, determine if the torque about the hinge is positive (+), negative (–), or zero (0).




13. Six forces, each of magnitude either $F$ or $2F$, are applied to a door. Rank in order, from largest to smallest, the six torques $\tau_1$ to $\tau_6$ about the hinge.




Order: $\tau_2 > \tau_1 = \tau_4 > \tau_3 = \tau_5 > \tau_6$

Explanation:

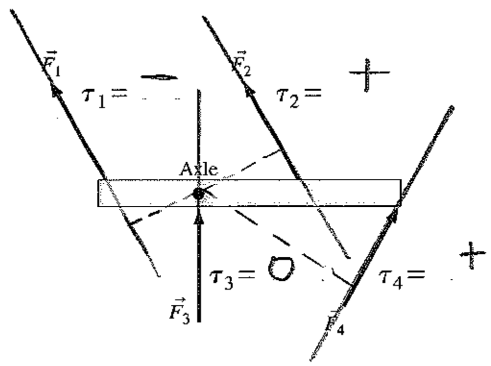
$\tau = r F_\perp$

$\tau_6 = 0$ since $F_\perp = 0$

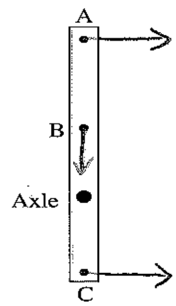
$\tau_1 = \frac{L}{4}(2F) = \frac{1}{2}LF$, $\tau_2 = L(2F) = 2LF$, $\tau_3 = \tau_5 = \frac{L}{2}F\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}LF$

$\tau_4 = \frac{L}{2}F = \frac{1}{2}LF$

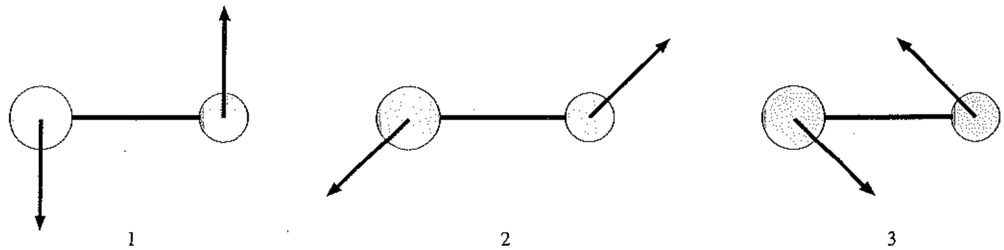
14. Four forces are applied to a rod that can pivot on an axle. For each,
    a. Use a **black** pen or pencil to draw the line of action.
    b. Use a **red** pen or pencil to draw and label the moment arm, or state that $r_\perp = 0$.
    c. Determine if the torque about the axle is positive (+), negative (–), or zero (0). black ——
    red - - -




15. a. Draw a force vector at A whose torque about the axle is negative.
    b. Draw a force vector at B whose torque about the axle is zero.
    c. Draw a force vector at C whose torque about the axle is positive.



16. The dumbbells below are all the same size, and the forces all have the same magnitude. Rank in order, from largest to smallest, the torques $\tau_1$, $\tau_2$, and $\tau_3$ about the midpoint of each connecting rod.

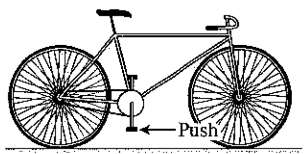



Order: $\tau_1 > \tau_2 = \tau_3$

Explanation:

The distance r is the same for all applied forces. $F_\perp$ is the same for 2 and 3. $F_\perp$ for 1 is larger than for 2 and 3.

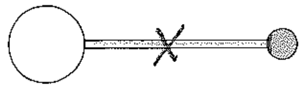
17. A bicycle is at rest on a smooth surface. A force is applied to the bottom pedal as shown. Does the bicycle roll forward (to the right), backward (to the left), or not at all? Explain.




Forward. The push applies a negative torque (clockwise) to the wheel, which pushes the ground to the left. By Newton's 3rd Law, the ground pushes the bike back towards the right, and the bike moves forward.

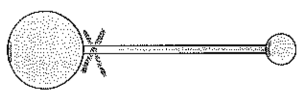
## 7.4 Gravitational Torque and the Center of Gravity

18. Two spheres are connected by a rigid but massless rod to form a dumbbell. The two spheres have equal mass. Mark the approximate location of the center of gravity with an ×. Explain your reasoning.



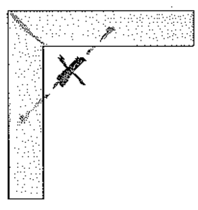
The spheres have equal mass so the center of gravity is halfway between their centers.

19. Two spheres are connected by a rigid but massless rod to form a dumbbell. The two spheres are made of the same material. Mark the approximate location of the center of gravity with an ×. Explain your reasoning.



The larger sphere is much more massive so the center of gravity is much closer to its center. (It could even be inside the larger sphere.)

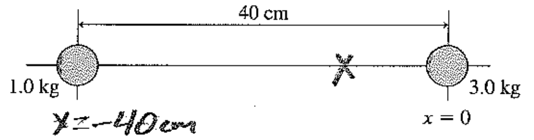
20. Mark the center of gravity of this object with an ×. Explain.



The object can be considered to be a composite of two bars. The center of gravity of the full object lies on a line connecting the two centers of gravity. By symmetry, it lies on the midpoint.
(This shape is similar to that of a boomerang, and the location of the center of gravity is the center of rotation of a boomerang when tossed.)

21. a. Find the coordinates for and *mark* the center of gravity for the pair of masses shown, using the center of the 3.0 kg mass as the origin.

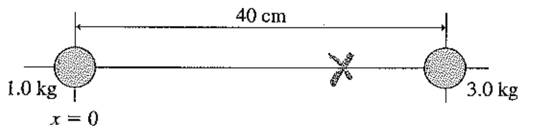
$$x_{cm} = \frac{(1.0\,kg)(-40\,cm) + (3.0\,kg)(0\,cm)}{(1+3)\,kg}$$




$x_{cg} =$ $-10$ cm

b. Find the coordinates for and *mark* the center of gravity for the pair of masses shown, using the center of the 1.0 kg mass as the origin.

$$x_{cm} = \frac{(1.0\,kg)(0\,cm) + (3.0\,kg)(40\,cm)}{(1+3)\,kg}$$

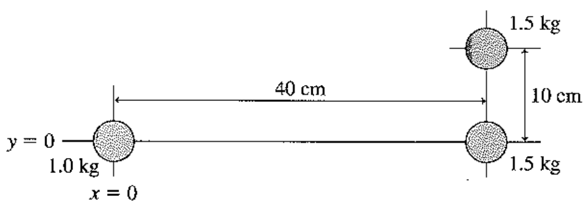



$x_{cg} =$ 30 cm

c. How do the locations for the marks in parts a and b compare? How do the coordinates compare?

Same location, but described in two different coordinate systems.

d. The 3.0 kg mass from parts a and b above is separated into two 1.5 kg pieces. One of these is moved 10 cm in the +*y*-direction. Find the coordinates for and *mark* the center of gravity of the new system using the center of the 1.0 kg mass as the origin.



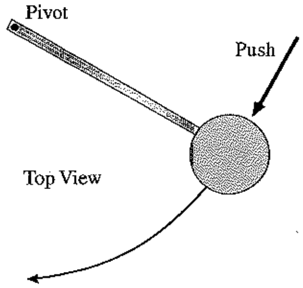

$x_{cg} =$ 30 cm (by comparison with above)

$y_{cg} =$ 0 cm (by symmetry)

e. What effect did separating the 3.0 kg mass along the *y*-direction have on the *x*-component of the center of gravity of the system?

No effect on either component. The x location of all masses did not change relative to the earlier problem. The masses were separated symmetrically in the y direction, so the overall effect was no change.

## 7.5 Rotational Dynamics and Moment of Inertia




22. A student gives a quick push to a ball at the end of a massless, rigid rod, causing the ball to rotate clockwise in a *horizontal* circle. The rod's pivot is frictionless.
   a. As the student is pushing, is the torque about the pivot positive, negative, or zero? negative
   b. After the push has ended, does the ball's angular velocity
      i. Steadily increase?
      ii. Increase for awhile, then hold steady?
      iii. Hold steady?
      iv. Decrease for awhile, then hold steady?
      v. Steadily decrease?

   Explain the reason for your choice.

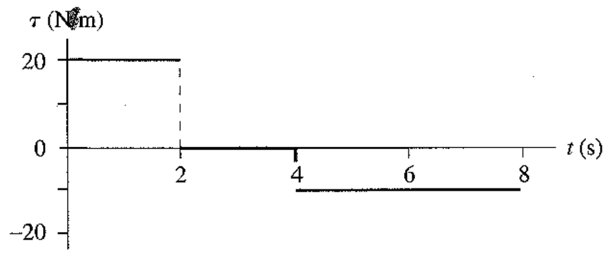
   A torque is necessary to change an object's rotation. No torques are present once the push has ended.

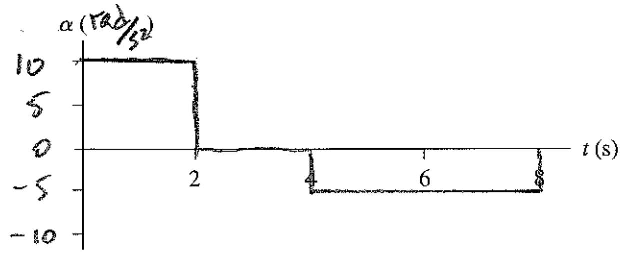
   c. Right after the push has ended, is the torque positive, negative, or zero? zero

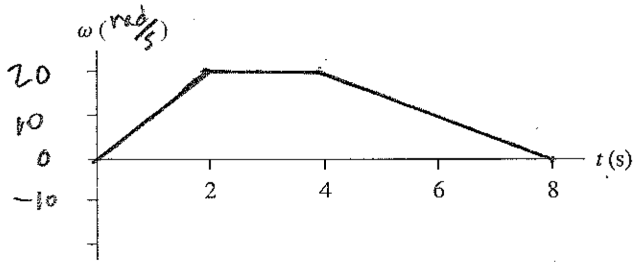
23. The top graph shows the torque on a rotating wheel as a function of time. The wheel's moment of inertia is 10 kg · m². Draw graphs of $\alpha$-versus-$t$ and $\omega$-versus-$t$, assuming $\omega_0 = 0$.

$I = 10\ kg\,m^2$

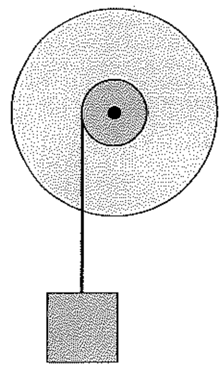
$\tau = I\alpha \Rightarrow \alpha = \tau/I$



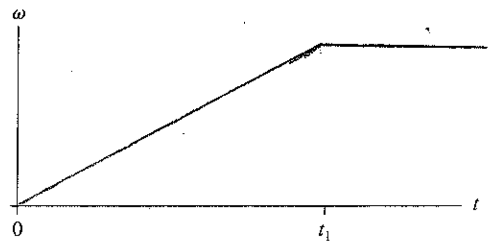



24. The wheel turns on a frictionless axle. A string wrapped around the smaller diameter shaft is tied to a block. The block is released at $t = 0$ s and hits the ground at $t = t_1$.



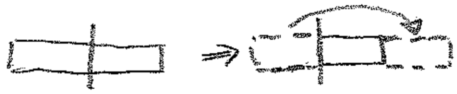
a. Draw a graph of $\omega$-versus-$t$ for the wheel, starting at $t = 0$ s and continuing to some time $t > t_1$.




b. Is the magnitude of the block's downward acceleration greater than $g$, less than $g$, or equal to $g$? Explain.

Less than g. The tension in the string is non-zero, so the block is not in free fall.

25. The moment of inertia of a uniform rod about an axis through its center is $\frac{1}{12}ML^2$. The moment of inertia about an axis at one end is $\frac{1}{3}ML^2$. Explain *why* the moment of inertia is larger about the end than about the center.

The center of the rod is, on average, closer to all points on the rod compared to an end, thus making $I = \Sigma mr^2$ smaller. Effectively, a section of the rod has been moved away from the rotation axis: 

26. You have two steel spheres. Sphere 2 has twice the radius of Sphere 1. By what *factor* does the moment of inertia $I_2$ of Sphere 2 exceed the moment of inertia $I_1$ of Sphere 1?

Since the volume of a sphere $V \propto r^3$, the mass of sphere 2 is $2^3 = 8$ times the mass of Sphere 1.

Since $I \propto mr^2$, $I_2 \propto (8m)(2r)^2 = 32 I_1$.

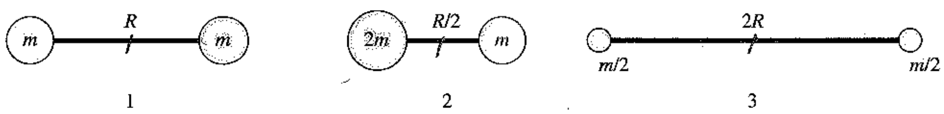
$$\frac{I_2}{I_1} = 32$$

27. The professor hands you two spheres. They have the same mass, the same radius, and the same exterior surface. The professor claims that one is a solid sphere and that the other is hollow. Can you determine which is which without cutting them open? If so, how? If not, why not?

It is easier to rotate the solid sphere because its moment of inertia is smaller than for the hollow sphere.

$$\left.\begin{aligned} &\text{Solid sphere: } I = \tfrac{2}{5} MR^2 \\ &\text{Hollow sphere: } I = \tfrac{2}{3} MR^2 \end{aligned}\right\} \tfrac{2}{5} < \tfrac{2}{3}$$

28. Rank in order, from largest to smallest, the moments of inertia $I_1$, $I_2$, and $I_3$ about the midpoint of each connecting rod.




Order: $I_3 > I_1 > I_2$

Explanation:

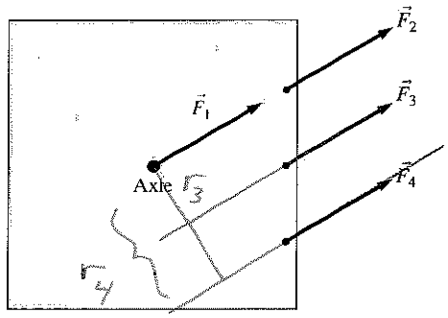
$$I_1 = 2\left[m\left(\frac{R}{2}\right)^2\right] = \frac{1}{2} MR^2$$

$$I_2 = 2m\left(\frac{R}{4}\right)^2 + m\left(\frac{R}{4}\right)^2 = \frac{3}{16} MR^2$$

$$I_3 = 2\left[\frac{m}{2} R^2\right] = MR^2$$

## 7.6 Using Newton's Second Law for Rotation

29. A square plate can rotate about an axle through its center. Four forces of equal magnitude are applied to different points on the plate. The forces turn as the plate rotates, maintaining the same orientation with respect to the plate. Rank in order, from largest to smallest, the angular accelerations $\alpha_1$ to $\alpha_4$.




Order: $\alpha_4 > \alpha_3 > \alpha_1 = \alpha_2$

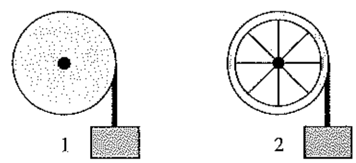
Explanation:

$\tau_1 = \tau_2 = 0$ since the lines of action for $\vec{F}_1$ and $\vec{F}_2$ pass through the axle.

$\tau_4 > \tau_3$ since $r_4 > r_3$.

$I$ of object is constant, and $\alpha = \tau/I$. Rank by $\tau$.

30. A solid cylinder and a cylindrical shell have the same mass, same radius, and turn on frictionless, horizontal axles. (The cylindrical shell has lightweight spokes connecting the shell to the axle.) A rope is wrapped around each cylinder and tied to a block. The blocks have the same mass and are held the same height above the ground. Both blocks are released simultaneously. The ropes do not slip.




Which block hits the ground first? Or is it a tie? Explain.

Block 1.

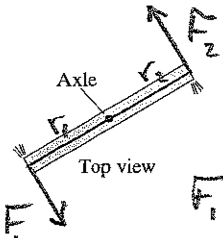
The weight of the block provides the force that both accelerates the block and causes a torque on the cylinders. The cylinder with the smaller rotational inertia will accelerate faster.

$I_{shell} = MR^2$

$I_{solid} = \frac{1}{2}MR^2$

31. A metal bar of mass $M$ and length $L$ can rotate in a horizontal plane about a vertical, frictionless axle through its center. A hollow channel down the bar allows compressed air (fed in at the axle) to spray out of two small holes at the ends of the bar, as shown. The bar is found to speed up to angular velocity $\omega$ in a time interval $\Delta t$, starting from rest. What force does each escaping jet of air exert on the bar?

PSS 7.1




$F_1 = F_2 = F$

$r_1 = r_2 = r = \frac{L}{2}$

a. On the figure, draw vectors to show the forces exerted on the bar. Then label the moment arms of each force.

b. The forces in your drawing exert a torque about the axles. Write an expression for each torque, and then add them to get the net torque. Your expression should be in terms of the unknown force $F$ and "known" quantities such as $M$, $L$, $g$, etc.

$$\tau_1 = \tau_2 = rF \qquad \tau_{net} = 2rF = LF$$

c. What is the moment of inertia of this bar about the axle? $\frac{1}{12} ML^2$

d. According to Newton's second law, the torque causes the bar to undergo an angular acceleration. Use your results from parts b and c to write an expression for the angular acceleration. Simplify the expression as much as possible.

$$\alpha = \frac{\tau_{net}}{I} = \frac{LF}{\frac{1}{12} ML^2} = \frac{12F}{ML}$$

e. You can now use rotational kinematics to write an expression for the bar's angular velocity after time $\Delta t$ has elapsed. Do so.

$$\alpha = \frac{\Delta\omega}{\Delta t} \Rightarrow \Delta\omega = \omega_f - \cancel{\omega_i}^{\,0} = \omega = \alpha\,\Delta t = \frac{12F\Delta t}{ML}$$

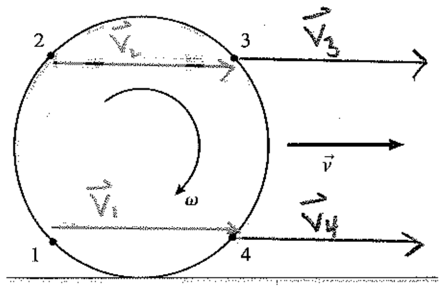
f. Finally, solve your equation in part e for the unknown force.

$$F = \frac{ML\omega}{12\,\Delta t}$$

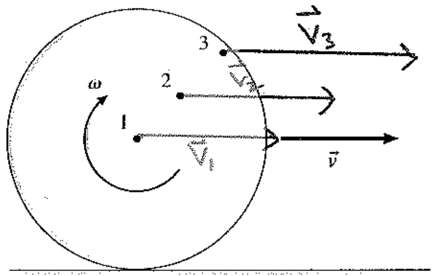
This is now a result you could use with experimental measurements to determine the size of the force exerted by the gas.

## 7.7 Rolling Motion

32. A wheel is rolling along a horizontal surface with the velocity shown. Draw the velocity vectors $\vec{v}_1$ to $\vec{v}_4$ at points 1 to 4 on the rim of the wheel.




33. A wheel is rolling along a horizontal surface with the velocity shown. Draw the velocity vectors $\vec{v}_1$ to $\vec{v}_3$ at points 1 to 3 on the wheel.




34. If a solid disk and a circular hoop of the same mass and radius are released from rest at the top of a ramp and allowed to roll to the bottom, the disk will get to the bottom first. *Without referring to equations*, explain why this is so.

The torque that causes both to accelerate down the ramp is provided by the weight of the objects applied at their centers and a moment arm from the point of contact with the surface. Both experience the same torque, but since the moment of inertia of the hoop is larger, it has a smaller angular acceleration, and the disk reaches the bottom first.